PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—January 2009

Define your terminology and explain your notation. If you require a standard result, state it before you use it; otherwise, give clear and complete proofs of your claims. Five problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Part I: Fourier Analysis

1. Let f be the function on $[-\pi, \pi]$ defined by $f(x) = 0$ for $-\pi \le x < 0$ and $f(x) = 1$ for $0 \le x < \pi$. Let g be the function on $[-\pi, \pi]$ defined by $g(x) = x^2$. You may assume that the Fourier series of f and g are as follows:

$$f(x) \sim \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \sin((2n+1)x)$$

and

$$g(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx.$$

Calculate $\int_{-\pi}^{\pi} f(x)g(x)\, dx$ using two methods: by direct integration, and by using the given Fourier series. Justify the second method.

2. In this problem we fix a function f that is 2π-periodic on the real line and integrable on $[-\pi, \pi]$. We write s_n for the nth partial sum of the Fourier series of f, and we define $\sigma_n = (s_0 + s_1 + \cdots + s_{n-1})/n$.

   (a) Give a sufficient condition for $\{s_n\}$ to converge to f at a point.

   (b) Give a sufficient condition for $\{\sigma_n\}$ to converge to f at a point.

   (c) Assume that f is continuous except at finitely many points. Suppose that the Fourier series of f converges pointwise to 0. What can you conclude about f? Be sure to prove your answer.

3. This problem concerns the Fourier (integral) transform of functions integrable on the real line $\mathbf{R}$ (that is, functions belonging to $L^1(\mathbf{R})$). We denote the Fourier transform of f by $\hat{f}$. We write S for the set of functions f on $\mathbf{R}$ with derivatives of all orders so that if k, n are nonnegative integers then $x^k f^{(n)}(x)$ is bounded, that is, there exists a constant C (depending on f, k, n) so that $|x^k f^{(n)}(x)| \le C$ for all x.

   (a) Explain why S is a subset of $L^1(\mathbf{R})$.

   (b) Let $f \in S$, and define g by $g(x) = xf(x)$. Explain why $g \in L^1(\mathbf{R})$, and state (without proof) a formula for the Fourier transform of g.

   (c) Use part (b) to show that if $f \in S$ then $\hat{f}$ is differentiable and the derivative of $\hat{f}$ is bounded.

   (d) Assume that $f \in S$. Explain why $\lambda \hat{f}(\lambda)$ is bounded. Sketch briefly how to expand this argument and the one in part (c) to show that $\hat{f} \in S$.

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Part II: Ordinary Differential Equations

1. Show that the following initial value problem has a unique local solution:

$$\frac{dy}{dx} = \frac{e^{-x}}{e^{x} + \sin^2 y} \qquad \text{for } x \geq 0,$$
$$y(0) = 1.$$

2. Consider the following system:

$$\begin{cases} \frac{dx}{dt} = -x^3 + xy^2 \\ \frac{dy}{dt} = -2x^2y - y^3. \end{cases}$$

   (a) Find all critical points and determine whether they are hyperbolic or not.

   (b) Use a Liapunov function of the form $V(x, y) = ax^2 + by^2$, where a and b are suitable constants, to show that the origin is asymptotically stable.

3. Consider the system of differential equations

$$(\mathrm{P}): \qquad \frac{d\boldsymbol{x}}{dt} = A(t)\boldsymbol{x},$$

where $A(t)$ is a real-valued $n \times n$ matrix which is continuous for $t \geq 0$. Let $X(t)$ be the fundamental matrix for (P), which is the $n \times n$ matrix so that $\frac{dX}{dt} = A(t)X$ and so that $X(0)$ is the identity matrix. Finally, for any $n \times n$ matrix M, we define the matrix norm $\|M\| = \sup_{\mathbf{v} \in \mathbf{R}^n, |\mathbf{v}|=1} |M\mathbf{v}|$, where $|\cdot|$ is the vector Euclidean norm.

   (a) Prove that $\|X(t)\| \leq e^{\int_0^t \|A(s)\| \, ds}$ for all $t \geq 0$.
   You may use the fact that $\|MN\| \leq \|M\| \|N\|$, where M and N are matrices.

   (b) Assume that $\int_0^\infty \|A(s)\| \, ds < \infty$. Prove that for every solution $\boldsymbol{x}(t)$ to (P) the limit $\lim_{t \to \infty} \boldsymbol{x}(t)$ exists. (*Hint: Consider the integral form.*)
   You may use the following result without proof: Given f continuous with $|f(s)| \leq |g(s)|$ for $0 \leq s < \infty$ and $\int_0^\infty |g(s)| \, ds < \infty$, then $\int_0^\infty f(s) \, ds$ converges.