PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—August 2008

Part I: Fourier Analysis

1. Consider the function f defined on the interval $-\pi < x \leq \pi$ by $f(x) = x$.

   (a) Find the (trigonometric) Fourier series of f.

   (b) Does the value at $x = \pi$ of the series in part (a) agree with $f(\pi)$? Explain.

   (c) Show that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots .$$

2. All functions considered in this problem are real-valued and square-integrable on an interval $[a, b]$, so they belong to $L^2([a, b])$. Let $\{\varphi_1, \ldots, \varphi_n, \ldots\}$ be an orthonormal system on $[a, b]$.

   (a) Fix a function f and define $c_n = \int_a^b f(x)\varphi_n(x)\, dx$ for each n. Show that $c_n \to 0$ as $n \to \infty$.

   (b) Suppose that there exists a function φ_0 not belonging to the given orthonormal system so that the enlarged system $\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}$ is still an orthonormal system. Is the original system complete?

   (c) Give an example of an orthonormal system on $[-\pi, \pi]$ and a function f for which the Bessel inequality is not an equality.

3. This problem concerns the Fourier (integral) transform of functions integrable on the real line $\mathbf{R}$ (that is, functions belonging to $L^1(\mathbf{R})$). We denote the Fourier transform of f by $\hat{f}$.

   (a) What can you say about $\lim_{\lambda \to \infty} \hat{f}(\lambda)$ if f is integrable? You do not need to prove your answer.

   (b) Is $\cos \lambda$ the Fourier transform of an integrable function?

   (c) State (without proof) a formula for the Fourier transform of f' under the assumption that f and f' are integrable.

   (d) Is $(1 + \lambda^2)^{-1/2}$ the Fourier transform of an integrable function that has an integrable derivative?

   (e) Is $e^{-\lambda^2}$ the Fourier transform of an integrable function?

   (f) Is the function defined to be 1 for $|\lambda| < 1$ and 0 for $|\lambda| \geq 1$ the Fourier transform of an integrable function?

Part II: Ordinary Differential Equations

August 2008

1. For $n = 0, 1, 2, \ldots$, the Legendre polynomials $P_n(x)$ are solutions to the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \qquad x \in [-1, 1].$$

   (a) Find all singular points (except ∞) of the Legendre equation. Determine whether they are weakly singular (regular singular) or strongly singular (irregular singular).

   (b) Show that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ for $m \neq n$.
   Hint: Write the equation in Sturm-Liouville form.

2. Consider the following damped pendulum problem for $x(t)$:

$$x'' + x' + \sin x = 0.$$

   (a) Define $y = x'$ and rewrite the above problem as a system of first order equations. Show that $(0,0)$ is an equilibrium point (also called a stationary point or a critical point).

   (b) Show that the following are both Liapunov functions for the system:

$$(1)\ V(x,y) = \frac{1}{2}y^2 + (1-\cos x) \qquad (2)\ V(x,y) = \frac{1}{2}(x+y)^2 + x^2 + \frac{1}{2}y^2.$$

   Is the origin stable, asymptotically stable, or unstable?
   Hint for (2): From Taylor's formula, we have $\sin x = x - \alpha x^3/3!$, where α depends on x and $0 < \alpha < 1$ for $-\pi/2 < x < \pi/2$. Reformulate $\dot{V}(x,y)$ in polar coordinates and prove the result for r small enough.

3. Consider the system of differential equations

$$(\mathrm{P}) \qquad \begin{cases} \frac{d\boldsymbol{x}}{dt} = (A + B(t))\boldsymbol{x} \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases}$$

   where A is a constant $n \times n$ matrix and $B(t)$ is an $n \times n$ matrix function which is continuous for $t \geq 0$.

   (a) Show that $\boldsymbol{x}(t)$ is a solution of (P) if and only if $\boldsymbol{x}(t)$ satisfies

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}B(\tau)\boldsymbol{x}(\tau)\,d\tau.$$

   (b) Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} e^{-t} & e^{-2t} \\ e^{-2t} & e^{-t} \end{pmatrix}.$$

   Use part (a) and Gronwall's inequality to show that $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$ for all initial values $\boldsymbol{x}_0$.