PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—January 2008

Part I: Fourier Analysis

1. Consider the function f defined on the interval $0 \leq x \leq \pi$ by $f(x) = x$.

   (a) Expand f in a Fourier cosine series.

   (b) At what points of the interval $-\pi \leq x \leq \pi$ does the cosine series from part (a) converge, and to what does it converge at those points? Explain.

   (c) At what points of the interval $-\pi \leq x \leq \pi$ does the sine series of f converge, and to what does it converge at those points? Note: You do not need to find the sine series, but you should be sure to explain your answer.

2. Assume that f is a 2π-periodic function that is integrable on $[-\pi, \pi]$ (that is, belongs to $L^1([-\pi, \pi])$). Write the Fourier series of f as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

   Assume that $|a_n| \leq 1/n^3$ and $|b_n| \leq 1/n^3$ for all $n \geq 1$.

   (a) Show that the Fourier series of f converges uniformly (to something).

   (b) For the rest of this problem assume also that f is continuous. Explain why the Fourier series of f must converge uniformly to f.

   (c) Show that the series obtained by differentiating the Fourier series of f term by term converges uniformly.

   (d) Show that f is differentiable.

3. Assume that f has a continuous derivative on $\mathbf{R}$ and that $f(x) = 0$ if $|x| \geq 1$. Let $g = f'$.

   (a) Derive (with justification) a formula for $\hat{g}$, the Fourier (integral) transform of g, in terms of $\hat{f}$.

   (b) Express $\int_{-\infty}^{\infty} |g(x)|^2 \, dx$ in terms of an integral involving $\hat{f}$.

Part II: Ordinary Differential Equations

1. Let p, q, r be continuous real-valued functions on $\mathbf{R}$ with $p > 0$. Prove that the equation
$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$
is equivalent to
$$(a(t)x'(t))' + b(t)x(t) = 0,$$
where a is continuously differentiable and b is continuous.

2. Consider the system $r' = r(r-1)(r-2)^2(3-r), \theta' = 1$ in polar coordinates. Find all periodic orbits and classify as to stability.

3. Prove that every solution $x(t)$ of the differential equation
$$\frac{dx}{dt} = x^2 - x^5, x(0) > 0,$$
satisfies $\lim_{t\to\infty} x(t) = 1$.