PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—August 2007

Part I: Fourier Analysis

1. Be sure to give a brief reason for your answer in each part.

   (a) At what points of the interval $[-\pi, \pi]$ does the trigonometric series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx$ converge?

   (b) At what points of the interval $[-\pi, \pi]$ does the trigonometric series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx$ converge?

   (c) Is the trigonometric series $\sum_{n=1}^{\infty} (-1)^n \sin nx$ the Fourier series of an integrable function on $[-\pi, \pi]$ (that is, a function in $L^1([-\pi, \pi])$)?

   (d) Is the trigonometric series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin nx$ the Fourier series of a square-integrable function on $[-\pi, \pi]$ (that is, a function in $L^2([-\pi, \pi])$)?

   (e) Is the trigonometric series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx$ the Fourier series of a 2π-periodic function that is differentiable on the real line?

2. Assume that f is an integrable function on $[-\pi, \pi]$ (that is, a function in $L^1([-\pi, \pi])$). Suppose that $f(x) = 0$ if $-1 \leq x \leq 1$. What can you say about the Fourier series of f? Be as specific as you can.

3. (a) Define the Fourier (integral) transform of a function that is integrable on the real line (that is, belongs to $L^1(\mathbb{R})$).

   (b) State the Fourier inversion theorem.

   (c) In this part, for ease of notation we denote the Fourier transform of a function g by $\mathcal{F}(g)$ instead of $\hat{g}$. Assume that both f and $\mathcal{F}(f)$ are integrable on the real line. Use part (b) to find a simple formula for $\mathcal{F}(\mathcal{F}(f))$ in terms of f.

Part II: Ordinary Differential Equations

1. Show that the differential equation $x' = 2x^2$ has no solution such that $x(0) = 2007$ and $x(t)$ is defined for all real numbers t.

2. Consider the system of differential equations

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2)$$

$$\frac{dy}{dt} = -x + y(1 - x^2 - y^2).$$

   (a) Show that there is a unique solution $(x(t), y(t))$ defined for all $t \in \mathbb{R}$ with initial condition $(x(0), y(0)) = (x_0, y_0)$.

   (b) Show that the solution referred to in part (a) approaches the circle $x^2+y^2 = 1$ as $t \to \infty$ provided $x_0 \neq 0$ and $y_0 \neq 0$.

3. Let $x(t) \in \mathbb{R}^3$ be a nontrivial solution to the system

$$\frac{dx}{dt} = Ax,$$

where $A = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}$. Prove that $\|x(t)\|$ is an increasing function of t.
(Note that $\|x(t)\|$ stands for the Euclidean norm of $x(t)$ in $\mathbb{R}^3$.)