PhD COMPREHENSIVE EXAM—FOURIER ANALYSIS & ODE—January 2006

Part I: Fourier Analysis

1. Let f be the function with period 2π so that $f(x) = x^3$ if $-\pi < x \leq \pi$. In this problem you may assume that the Fourier series for f on $[-\pi, \pi]$ is

$$2\pi^2 \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} + 12\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^3}.$$

   (a) Integrate to find the Fourier series for x^4 on $[-\pi, \pi]$. Explain briefly why this procedure is valid.

   (b) At what points of $[-\pi, \pi]$ does the Fourier series of f converge to f? At what points of $[-\pi, \pi]$ does the Fourier series from part (a) converge to x^4? Give brief explanations for your answers.

2. All functions considered in this problem are real-valued and square-integrable on a fixed interval $[a, b]$. For such a function f we define $||f||$ as the nonnegative square root of $\int_a^b f^2(x)\, dx$.

   (a) Let $\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\}$ be an orthonormal system on $[a, b]$, and let $\{c_0, c_1, \ldots\}$ be an infinite sequence of real numbers. Show that, for all $N \geq 1$,

$$\left\|\sum_{n=0}^{N} c_n\varphi_n\right\|^2 = \sum_{n=0}^{N} c_n^2.$$

   (b) Define what it means for an orthonormal system to be complete, and give an equivalent formulation of completeness.

3. In this problem we assume that f is a function with period 2π whose first and second derivatives are continuous on the real line. The goal is to show, without making use of the Dirichlet kernel, that the Fourier series of f converges uniformly to f on the real line.

   (a) Use the definitions to show that if the trigonometric Fourier series for f is

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

   and the trigonometric Fourier series for f' is

$$f'(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

   then $A_n = -b_n/n$ and $B_n = a_n/n$ when $n \geq 1$. Use this to find a similar relation between the Fourier coefficients of f (namely, the A_n and B_n) and those of f''.

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(b) What does the Riemann-Lebesgue lemma say about the Fourier coefficients of an absolutely integrable function h?

(c) Use parts (a) and (b) to show that the Fourier series of f converges uniformly (to something) on the real line.

(d) Consider the following general statement: "If F is 2π-periodic on the real line and absolutely integrable on $[-\pi, \pi]$, and if also F is ________________ at a point x_0, then if the Fourier series of F converges at x_0 it must converge to $F(x_0)$." Fill in the blank in this statement (with a term such as continuous or differentiable) and give a very brief explanation for your answer.

(e) Explain from parts (c) and (d) why the Fourier series of f converges uniformly to f on the real line.

Part II: Ordinary Differential Equations

1. (a) State an existence and uniqueness theorem for the following initial value problem (IVP):

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

   (b) Find at least two solutions of the IVP

$$\frac{dy}{dx} = \sqrt{|y|}, \quad y(0) = 0.$$

   Explain why the theorem stated in part (a) does not apply to this IVP.

   (c) Show that the IVP

$$\frac{dy}{dx} = -y - y^3 - y^7, \quad y(0) = 1,$$

   has a solution for all $x > 0$.

2. Consider the system of differential equations

$$\frac{dx}{dt} = -y + xq(x, y), \quad \frac{dy}{dt} = x + yq(x, y), \tag{1}$$

where $q(x, y) = -(x^6 + y^6)$.

   (a) Classify the critical point (0,0) as to type, and determine whether it is stable, asymptotically stable, or unstable for the linearized system around (0,0).

   (b) Is the system (1) almost linear? What conclusion can you draw about the nonlinear system at (0,0)?

   (c) Use Liapunov's second method to determine whether (0,0) is stable, asymptotically stable, or unstable.

3. (a) State one version of Gronwall's lemma.

   (b) Consider the system of differential equations

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{g}(t, \mathbf{y}). \tag{2}$$

   Here A is a constant $n \times n$ matrix, and $\mathbf{g}(t, \mathbf{y})$ is continuous for $t \geq 0$ and all $\mathbf{y}$. Assume that every eigenvalue of A has negative real part, and that

$$\lim_{|\mathbf{z}| \to 0} \frac{|\mathbf{g}(t, \mathbf{z})|}{|\mathbf{z}|} = 0 \quad \text{uniformly for } 0 \leq t < \infty.$$

   In particular, $\mathbf{g}(t, \mathbf{0}) = \mathbf{0}$ for $t \geq 0$. Show that the zero solution of (2) is asymptotically stable. You may use without proof the fact that

$$|e^{At}| \leq Ce^{-\epsilon t} \quad \text{for} \quad t \geq 0$$

   for some constants $C > 0$ and $\epsilon > 0$. Here $|e^{At}|$ denotes a matrix norm of e^{At}. (The inequality is valid for any one of the matrix norms.)