Ph.D. Comprehensive Exam–ODE & Fourier Analysis

August 2005

Part I: Fourier Analysis

1. Let f be the function on $[-\pi, \pi]$ defined by the condition that f is odd and $f(x) = \pi - x$ if $0 < x \leq \pi$.

   (a) Find the (trigonometric) Fourier series of f.

   (b) At what points of the interval $[-\pi, \pi]$ does the Fourier series of f converge? To what does the series converge at those points? Be sure to explain your answer.

   (c) Does the Fourier series converge uniformly on $[-\pi, \pi]$? Be sure to explain your answer.

   (d) Let g be the function on $[-\pi, \pi]$ defined by $g(x) = x^8$. Does the (trigonometric) Fourier series of g converge uniformly to g on the interval $[-\pi, \pi]$? Note: You do not need to find the Fourier series of g, but you should be sure to explain your answer.

2. Suppose that f has period 2π and is absolutely integrable on $[-\pi, \pi]$. Let the Fourier series for f be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

   Suppose that $\sum_{n=1}^{\infty}(|a_n| + |b_n|)$ converges. What can you conclude about the convergence of the Fourier series for f? Prove your result.

3. Recall that the Fourier transform $\hat{f}$ of an absolutely integrable function f on the real line $\mathbf{R}$ is defined for $\lambda \in \mathbf{R}$ by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx.$$

   (a) Let f be absolutely integrable on $\mathbf{R}$ and c a positive constant. Define g on $\mathbf{R}$ by $g(x) = f(x/c)$. Derive a formula relating $\hat{g}$ to $\hat{f}$.

   (b) State a version of the Fourier inversion theorem (also called the Fourier integral theorem) for functions absolutely integrable on $\mathbf{R}$.

Part II: Ordinary Differential Equations

1. Consider the initial value problem

$$y' = \frac{1}{x^2 + y^2}, \qquad y(0) = y_0 \neq 0.$$

   (a) Prove that the solution of this problem are all defined over $(-\infty, \infty)$.

   (b) Prove that $\lim_{x \to \pm\infty} y(x)$ exists and is finite.

2. Consider the Sturm-Liouville eigenvalue problem

$$\begin{cases} (p(x)y'(x))' + (q(x) + \lambda)y(x) = 0, & x \in [a, b], \\ \alpha_1 y(a) + \alpha_2 y'(a) = 0, & \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases} \tag{1}$$

   where $p \in C^1[a, b]$, $p(x) > 0$ for all $x \in [a, b]$ and q is a continuous function on $[a, b]$. If

$$\alpha_1\alpha_2 = \beta_1\beta_2 = 0, \qquad m = \min_{x \in [a,b]} (-q(x)),$$

   show that the eigenvalues of (1) have lower bound m, namely, $\lambda \geq m$ for any eigenvalue of (1).

3. Let $(0, 0)$ be a stationary point of the system

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y). \end{cases}$$

   Suppose $L(x, y)$ is a Lyapunov function for $(0, 0)$ (namely L is positive definite near $(0, 0)$) with $\dot{L}(x, y) \equiv \nabla L \cdot (f, g)$ being positive in a neighborhood of $(0, 0)$. Prove that $(0, 0)$ is not stable.