Comprehensive Exam–Numerical Analysis

August 2009

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider an $M \times M$ uniform grid on the unit square $(0,1) \times (0,1)$, where M is a positive integer. Let $h = 1/M$ denote the step size. On each interior grid point $(x_m, y_n) = (mh, nh)$, for $1 \le m, n \le M-1$, define the difference operator ∇_h^2 by

$$\nabla_h^2 v_{m,n} \equiv \frac{v_{m+1,n} - 2v_{m,n} + v_{m-1,n}}{h^2} + \frac{v_{m,n+1} - 2v_{m,n} + v_{m,n-1}}{h^2}.$$

Here $v_{m,n}$ is the grid function and we assume it is 0 on all boundary nodes. Prove that there exists a positive constant C, independent of h, such that

$$\max_{1 \le m,n \le M-1} |v_{m,n}| \le C \max_{1 \le m,n \le M-1} |\nabla_h^2 v_{m,n}|.$$

(For this problem, you may use the discrete maximum principle without proof.)

2. Consider solving the following initial value problem:

$$\begin{aligned} u_t &= u_x, \qquad (t,x) \in (0,\infty) \times (-\infty,\infty), \\ u(0,x) &= u_0(x), \quad x \in (-\infty,\infty), \end{aligned}$$

with the following finite difference scheme: for $h, k > 0$,

$$\begin{aligned} \frac{v_m^{n+1} - v_m^n}{k} &= \frac{v_m^{n+1} + v_m^n - v_{m-1}^{n+1} - v_{m-1}^n}{2h}, m \in \mathbb{Z}, n \in \{0\} \cup \mathbb{Z}_+, \\ v_m^0 &= u_0(mh), \quad m \in \mathbb{Z}, \end{aligned}$$

where u_0 is continuous on $\mathbf{R}$.
(a) Show that the scheme is consistent.
(b) Find the order of accuracy of the scheme. Justify your answers.

3. Suppose $f \in C^\infty([0,1])$ and consider the boundary value problem:

$$-u'' = f,\ x \in (0,1),\ u(0) = u(1) = 0.$$

Consider solving the problem using the following scheme:

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{(\triangle x)^2} = f(j\triangle x),\ j = 1, 2, \cdots, J-1; U_0 = 0, U_J = 0, \quad (1)$$

where $\triangle x = \frac{1}{J}$. Show that there exists a unique solution $\mathbf{U} = [U_1, \cdots, U_{J-1}]^T$ satisfying (1).

4. Let

$$A = \begin{bmatrix} -11/5 & 23/5 \\ -19/5 & 17/5 \end{bmatrix} \in \mathbf{R}^{2\times 2}.$$

Determine $\|A\|_2$, the 2-norm of A. Hint: Show the (column-) vectors $\mathbf{u} = [1,1]^T$ and $\mathbf{v} = [-1,1]^T$ are eigenvectors of the matrix $B = AA^T$. Do not confuse the 2-norm of matrix A with its Frobenius norm.

5. Let $A \in \mathbf{R}^{n\times n}$ be a symmetric matrix. Suppose that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are the orthonormal eigenvectors with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $\mathbf{q} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in \mathbf{R}^n$ be a vector such that $\|\mathbf{q}\|_2 = 1$.

   (a) Derive an expression for the Rayleigh quotient $\rho = \mathbf{q}^T A\mathbf{q}$ in terms of the coefficients c_i and the eigenvalues λ_i.

   (b) Show that $|\lambda_1 - \rho| \le C\,\|\mathbf{v}_1 - \mathbf{q}\|^2$, where $C = \max_{2\le k\le n} |\lambda_1 - \lambda_k|$.

6. Suppose $\|\cdot\|$ is a matrix norm in $\mathbf{R}^{n\times n}$ such that $\|XY\| \le \|X\|\,\|Y\|$, for all $X, Y \in \mathbf{R}^{n\times n}$. Suppose A and E are $n\times n$ real matrices, A is invertible and $\|A^{-1}E\| < 1$. Prove the following:

   (a) $A - E$ is invertible.
   (b) $\|(A-E)^{-1}\| \le \frac{\|A^{-1}\|}{1-\|A^{-1}E\|}$.