Comprehensive Exam–Numerical Analysis

January 2009

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. In order to approximate the solution of

$$y' = y \quad \text{with } y(0) = 1,$$

the following form of Taylor method is applied :

$$y_{n+1} = y_n + hy_n, \text{ where } x_n = n\,h \text{ and } y_0 = 1.$$

Show that $\lim_{h\to 0} y_N = e$, where $Nh = 1$ and y_N is the approximation to $y(1)$.

2. Consider the following initial value problem:

$$\begin{aligned} u_t + u_x &= 0, \quad x \in \mathbf{R}^1, \quad t > 0, \\ u(t,x) &= u_0(x), \quad x \in \mathbf{R}^1, \end{aligned}$$

where $u_0(x) \in C^\infty([0,1])$.

(i) Propose the backward-time central-space scheme. Use k and h for time step and space step respectively. Use $v_m^n = u(nk, mh)$.

(ii) Prove that the above scheme is unconditionally stable in the 2-norm. The 2-norm of v^n is given

$$\|v^n\|_2 = \left(h \sum_{j=-\infty}^{\infty} |v_j^n|^2 \right)^{1/2}.$$

3. Suppose $f \in C^\infty([0,1])$ and consider the boundary problem:

$$-u'' = f, \qquad x \in (0,1), \qquad u(0) = u(1) = 0.$$

Consider solving the problem using the following scheme:

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{(\triangle x)^2} = f(j\triangle x), \; j = 1, 2, ..., J-1; U_0 = 0, U_J = 0, \qquad (1)$$

where $\triangle x = \frac{1}{J}$. Show that there exists a unique solution $U = [U_1, U_2, ..., U_{J-1}]^T$ satisfying (1).

4. Let $A \in \mathbf{R}^{n \times n}$ be an invertible matrix and let $b \in \mathbf{R}^n$ and $b \neq 0$. Let $\kappa(A)$ be the condition number of A, namely $\kappa(A) = \|A\| \|A^{-1}\|$. Let $\Delta A \in \mathbf{R}^{n \times n}$ and $\Delta b \in \mathbf{R}^n$. Suppose $\|\Delta A\| \leq \epsilon \|A\|$, $\|\Delta b\| \leq \epsilon \|b\|$,

$$A\,x = b \quad \text{and} \quad (A + \Delta A)\,y = b + \Delta b.$$

If $r \equiv \epsilon\, \kappa(A) < 1$, show that $A + \Delta A$ is invertible and

$$\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}.$$

5. Please calculate by hand. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

   (a) Find the singular value decomposition of A.

   (b) Find the least square solution of $Ax = b$.

6. Let $A \in \mathbf{R}^{n \times n}$ and let v be an eigenvector of A with associated eigenvalue λ. Assume $\|v\|_\infty = 1$, where $\|v\|_\infty = \max_{1 \leq j \leq n} |v_j|$ denotes the l_∞-norm of v. Let p be a number such that $(A - pI)^{-1}$ exists and $|\lambda - p| < |\mu - p|$ for any other eigenvalue μ of A.

   (a) Show that v is an eigenvector of $(A - pI)^{-1}$. What is the corresponding eigenvalue?

   (b) Assume that the span of all eigenvectors of A is $\mathbf{R}^n$. Show that, for any starting vector $y^{(0)}$ not perpendicular to v, the sequence $\{y^{(k)}\}$ generated by the algorithm

$$y_*^{(k+1)} = (A - pI)^{-1}\, y^{(k)}, \quad y^{(k+1)} = y_*^{(k+1)} / \|y_*^{(k+1)}\|_\infty$$

   converges to $\pm\, v$ in l_∞.

   (c) Explain how to obtain an approximation of λ using $\{y^{(k)}\}$.