Numerical Analysis Comprehensive Exam, January, 2007

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Note: *There are 2 hours and 6 problems in total. Solve all 6 problems. Brief details should be provided justifying each step towards final answers to all the problems.*

1. Consider solving the initial value problem:

$$u_t + u_x = 0, \quad x \in \mathbf{R}^1,\ t > 0, \quad u(x,0) = u_0(x), \quad x \in \mathbf{R}^1,$$

with the following finite difference scheme:

$$\frac{2U_j^{n+1} - U_{j+1}^n - U_{j-1}^n}{\Delta t} + \frac{U_{j+1}^n - U_{j-1}^n}{\Delta x} = 0, \quad U_j^0 = u_0(j\Delta x),$$

where $n = 1, 2, \cdots$, $j = 0, \pm 1, \pm 2 \cdots$ and Δt, $\Delta x > 0$.

(a) State briefly the definition of consistency of the above numerical scheme for the initial value problem. Suppose $\Delta t = (\Delta x)^\alpha$ with $\alpha > 0$. Find, and justify, the *exact* range of α so that the above numerical scheme is consistent.

(b) State briefly the definition of stability, in the 2-norm, of the above numerical scheme. Define $\nu \equiv \frac{\Delta t}{\Delta x}$ and the 2-norm of U^n as following:

$$\|U^n\|_2 \equiv \left[\Delta x \sum_{j=-\infty}^{+\infty} |U_j^n|^2\right]^{\frac{1}{2}}.$$

Find, and justify, the *exact* range of ν so that the above numerical scheme is stable in 2-norm.

2. Consider the following finite difference scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}.$$

where $n = 1, 2, \cdots$, $j = 0, \pm 1, \pm 2 \cdots$ and Δt, $\Delta x > 0$. Define:

$$\mu := \frac{\Delta t}{(\Delta x)^2}, \quad \|U^n\|_\infty := \sup_{-\infty < j < +\infty} |U_j^n|.$$

Find, and justify, the *exact* range of μ so that $\|U^n\|_\infty \le \|U^0\|_\infty$ for all $n \ge 1$ and for all U^0 with $\|U^0\|_\infty$ being finite.

3. Suppose $f \in C([0,1])$ and u solves the boundary value problem:

$$u_{xx}(x) = f(x), \quad x \in [0,1]; \quad u(0) = 0, \; u(1) = 0.$$

Consider solving the problem using the following scheme:

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{(\Delta x)^2} = f(j\Delta x), \quad j = 1, 2, \ldots, J-1; \quad U_0 = 0, \; U_J = 0,$$

where $\Delta x = 1/J$. Prove the following:

$$\lim_{\Delta x \to 0} \max_{0 \le j \le J} |U_j - u(x_j)| = 0.$$

4. (a) Define the Householder reflection $H \in \mathbf{C}^{4\times 4}$ by

$$H = I - \frac{2}{\mathbf{v}^*\mathbf{v}}\mathbf{v}\mathbf{v}^* \in \mathbf{C}^{4\times 4}$$

where the (column-) vector $\mathbf{v} = [1, 1, -1, 1]^T \in \mathbf{C}^4$ is given. Show that H is

(i) a *unitary* matrix,

(ii) a *hermitian* matrix.

You may wish to begin by stating definitions of conditions (i) and (ii) for the general case of a matrix $A \in \mathbf{C}^{n\times n}$.

(b) Show, if we let

$$A = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 2 & 5 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \in \mathbf{C}^{4\times 3},$$

then the matrix product HA is *upper triangular.*

5. Let

$$A = \begin{bmatrix} -11/5 & 23/5 \\ -19/5 & 17/5 \end{bmatrix} \in \mathbf{R}^{2\times 2}.$$

Determine $\|A\|_2$, the 2-norm of A. *Hint*: Show the (column-) vectors $\mathbf{u} = [1, 1]^T$ and $\mathbf{v} = [-1, 1]^T$ are eigenvectors of matrix $B = AA^T$. Do *not* confuse the 2-norm of matrix A with its *Frobenius* norm.

6. Suppose that for a given diagonalizable real matrix $A \in \mathbf{R}^{n\times n}$ with n distinct eigenvalues λ_i, $i = 1, \cdots, n$, we have obtained a number $\mu \approx \lambda_n$ by numerical computations.

(a) Briefly decribe the algorithm of *inverse iteration* for computing approximate eigenvectors, $\mathbf{v}^{(k)}$ of matrix A, such that $A\mathbf{v}^{(k)} \approx \lambda_n \mathbf{v}^{(k)}$, by solving systems of equations involving the approximate eigenvalue μ.

(b) Sketch a proof that inverse iteration will produce vectors $\mathbf{v}^{(k)}$, $k = 1, 2, \cdots$, converging to a suitable eigenvector $\mathbf{v}_n$, $A\mathbf{v}_n = \lambda_n \mathbf{v}_n$, $A\mathbf{v}^{(k)} \approx \lambda_n \mathbf{v}^{(k)}$, as $k \to \infty$, stating any additional hypotheses that may be required.