Numerical Analysis Comprehensive Exam

August, 2006

Note: *There are 2 hours and 6 problems in total. Solve all the 6 problems. Brief details should be provided justifying each step towards final answers to all the problems.*

1. Suppose u is the solution to the initial boundary value problem:

$$\partial_t u = \partial_x^2 u, \quad x \in [0,1], \; t \in [0,1],$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,1],$$

$$u(x,0) = u_0(x), \quad x \in [0,1],$$

where $u = u(x,t)$ is a scalar real function and $\partial_x^4 u \in C([0,1] \times [0,1])$. Consider the following finite difference scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^{n+1} + U_{j-1}^n}{(\Delta x)^2}, \quad n = 0,\ldots,N-1, \; j = 1,\ldots,J-1,$$

$$U_0^n = U_J^n = 0, \quad n = 0,\ldots,N, \quad U_j^0 = u_0(j\Delta x), \quad j = 0,\ldots,J,$$

where $\Delta x = 1/J$ and $\Delta t = 1/N$. Derive the following:

   (a) The truncation error of the scheme and the exact condition for the scheme to be consistent.

   (b) The exact condition for the scheme to be stable in L^2 norm.

2. Consider the following finite difference scheme.

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{U_j^{n+1} - U_{j-1}^{n+1}}{\Delta x} = 0, \quad n = 0,\ldots,N-1, \quad j = 1,\ldots,J,$$

$$U_0^n = 0, \quad n = 0,\ldots,N, \quad U_j^0 = u_0(j\Delta x), \quad j = 0,\ldots,J,$$

where $\Delta t = 1/N$ and $\Delta x = 1/J$. Derive the exact condition for the scheme to be stable in the maximum norm.

3. Suppose $u \in C^4([0,1])$ is the solution to the boundary value problem:

$$u_{xx}(x) = f(x), \quad x \in [0,1]; \quad u(0) = 0, \; u(1) = 1.$$

Consider solving the above problem using the following finite difference scheme:

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{(\Delta x)^2} = f(j\Delta x), \quad j = 1, 2, \ldots, J-1; \quad U_0 = 0, \; U_J = 1,$$

where $\Delta x = 1/J$. Derive an upper bound of $\max_{0 \le j \le J} |U_j - u(x_j)|$ in the order of $(\Delta x)^2$.

4. Calculate the Choleski-factorization $A = L\,L^T$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

5. Suppose $\|\cdot\|$ is a matrix norm in $\mathbf{R}^{n \times n}$ such that $\|XY\| \le \|X\|\|Y\|$, $\forall X, Y \in \mathbf{R}^{n \times n}$. Suppose A and E are $n \times n$ real matrices, A is invertible and $\|A^{-1}E\| < 1$. Prove the following:

   (a) $A - E$ is invertible.

   (b) $\|(A-E)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}E\|}$.

6. Suppose H is an upper Hessenberg matrix H. One QR-iteration step applied to H proceeds along these lines:

$$H = QR, \quad T \leftarrow RQ, \quad Q = U_1 U_2 \ldots U_{n-1}, \quad U_k = I - \left(\frac{2}{\mathbf{v}_k^* \mathbf{v}_k} \right) \mathbf{v}_k \mathbf{v}_k^*,$$

where each U_k is a Householder reflector. Prove the following:

   (a) For properly chosen U_k's, Q is upper Hessenberg.

   (b) For Q as obtained in (a), RQ is upper Hessenberg.