Numerical Analysis Comprehensive Exam

January, 2006

Notice: There are 2 hours and 6 problems in total. You should try to solve all the 6 problems. In problems 1-3, ignore the issue of possible numerical rounding errors involved when solving the finite difference schemes or when evaluating a given function. In problem 4, the possible numerical rounding errors should be taken into account.

1. Given the following initial boundary value problem:

$$\partial_t u = \partial_x^2 u + u, \quad x \in (0,1), \quad t \in (0,1],$$
$$u(0,t) = u(1,t) = 0, \quad t \in [0,1],$$
$$u(x,0) = u_0(x), \quad x \in [0,1].$$

where $u = u(x,t)$ is a scalar real function and $u_0 \in C^\infty([0,1])$. Consider the following finite difference scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2} + U_j^n,$$
$$U_j^0 = u_0(j\Delta x),$$

where $n = 0, 1, 2, \ldots, N$ and $j = 0, 1, \ldots, J$ are the usual super-index and sub-index, while $\Delta x = 1/J$ and $\Delta t = 1/N$.

   (a) Find a bound of the truncation error for this scheme.

   (b) Derive an estimate of $\max_{0 \leqslant j \leqslant J} |u(j\Delta x, 1) - U_j^N|$.

2. Given the following initial value problem:

$$\partial_t u + a\partial_x u + bu = 0, \quad x \in \mathbb{R}^1, \quad t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^1,$$

where a and b are real constants and $u = u(x,t)$ is a scalar real function.

Consider the following finite difference scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j-1}^n - U_{j+1}^n}{2\Delta x} + b\frac{U_{j-2}^n + U_{j+2}^n}{2} = 0,$$
$$U_j^0 = u_0(j\Delta x),$$

where $n = 0, 1, 2, \ldots$, $j = 0, \pm 1, \pm 2, \ldots$ and the ratio $\frac{\Delta t}{\Delta x}$ is assumed to be a constant.

(a) State briefly the definition of the **CFL condition** and then derive the CFL condition for the above scheme.

(b) Assume that $u_0 \in C^\infty(\mathbb{R}^1)$ and it is compactly supported. Use Fourier Analysis to study the convergence of this scheme (in L^2 norm) as $\Delta x \to 0$.

3. Given the following mixed boundary value problem

$$u_{xx} = f(x), \quad x \in (0,1),$$
$$u(0) = u_x(1) = 0,$$

assuming that the solution $u \in C^\infty([0,1])$. Consider solving the above problem using the following finite difference scheme:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{(\Delta x)^2} = f(j\Delta x), \quad j = 1, 2, \ldots, J-1;$$
$$U_J = U_{J-1}, \quad U_0 = 0,$$

where $\Delta x = 1/J$. Using the maximum principle to derive an estimate of

$$\max_{0 \leqslant j \leqslant J} |u(j\Delta x) - U_j|.$$

4. Scaling of linear equations is often necessary for accurate solution. For example, in the 2x2 system $Ax = b$, where the matrix $A = \begin{bmatrix} 1 & 1 \\ 10 & 10^5 \end{bmatrix}$, we divide by 10^5 in the second equation, to produce an equivalent system $Bx = d$ with the new matrix $B = \begin{bmatrix} 1 & 1 \\ 10^{-4} & 1 \end{bmatrix}$. Based on a theory of errors, we can readily give a standard, quantitative explanation of why solution of the second system is more accurate. What is that standard explanation?

5. Suppose A is a 6×6 matrix with eigenvalues 20, 19, 18, 17, 16 and 15. Let the corresponding eigenvectors be $\mathbf{x}_j$, $1 \leq j \leq 6$, respectively (i.e. $\mathbf{x}_1$ corresponds to 20). Let $\mathbf{v}_0 = c_1\mathbf{x}_1 + \cdots + c_6\mathbf{x}_6$ be any vector with $c_1 = 1$.

(a) In general, at what rate will the powers $\frac{1}{20^n} A^n \mathbf{v}_0$, $n = 1, 2, 3, \ldots$, converge to $\mathbf{x}_1$?

(b) What are the eigenvalues of the matrix $A - \sigma I$ for any real constant σ?

(c) Determine the optimum shift σ so that the powers $\frac{1}{(20-\sigma)^n}(A - \sigma I)^n \mathbf{v}_0$, $n = 1, 2, 3, \ldots$, converge to $\mathbf{x}_1$ with the fastest rate of convergence. Explain precisely why this choice is optimal.

6. (a) Define the term Householder matrix.

(b) Use Householder reduction to determine an orthogonal matrix P such that

$$P \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix} P^T$$

is tridiagonal.