Comprehensive Exam–Numerical Analysis

January 2017

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

Notation: In problems 1-2, denote by v_m^n the value of a real grid function v at point $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$ and $n \in \{0\} \cup \mathbb{Z}^+$, where $k, h > 0$.

1. Find the *exact* range of $\alpha \in \mathbb{R}$ such that the following finite difference scheme, for integer m and non-negative integer n,

$$v_m^{n+1} = v_m^{n-1} + \alpha(v_{m-1}^n - v_{m+1}^n),$$

is stable in L^2 norm. *Prove your claim.*

2. Consider the following finite difference scheme

$$v_m^{n+1} = (1 - 2b\mu)v_m^n + b\mu(1 - \alpha)v_{m+1}^n + b\mu(1 + \alpha)v_{m-1}^n,$$

where a, b are positive constants, $\mu = \frac{k}{h^2}$, $\alpha = \frac{ha}{2b}$ and $2b\mu \leq 1$. Find the *exact* range of α such that

$$\sup_m |v_m^{n+1}| \leq \sup_m |v_m^n|,$$

Prove your claim.

3. Let $N \geq 5$ be an integer. Consider the following finite difference scheme:

$$\frac{v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} - 4v_{m,n}}{h^2} + \frac{v_{m,n} - v_{m-1,n}}{h} = f_{m,n}, \quad m, n \in \{1, \ldots, N-1\},$$

and $v_{m,n} = g_{m,n}$ if m or n is 0 or N. Prove that for any given set of $f_{m,n}$'s and $g_{m,n}$'s, there exists one and only one grid function $\{v_{m,n}\}$ $(m, n \in \{1, \ldots, N-1\})$ as the solution of the finite difference scheme.

4. Let $a(u, v) = \int_0^1 u'(x)v'(x)dx$ and $V = \{v \in L_2(0,1) : a(v, v) < \infty\}$. Suppose $f \in C^0([0,1])$ and $u \in C^2([0,1])$ satisfies

$$a(u, v) = \int_0^1 f(x)v(x)dx, \text{ for all } v \in V.$$

Show that $u'' = -f$, $u'(1) = 0$ and $u'(0) = 0$.

5. (a) State the Lax-Milgram Theorem.

   (b) Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary and $V = H_0^1(\Omega)$. Let

$$a(u, v) = (\nabla u, \nabla v) \text{ for } u, v \in V,$$

and let $f \in L_2(\Omega)$. Use the Lax-Milgram Theorem to show that

$$a(u, v) = (f, v), \quad \forall v \in V$$

has a unique solution $u \in V$.

6. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary and $V = H_0^1(\Omega)$. Let $s \geq 0$ and let

$$a(u, v) = (\nabla u, \nabla v) + (u, v), \text{ for } u, v \in V.$$

Assume that there is a unique solution, u, to the variational problem

$$a(u, v) = (f, v), \text{ for all } v \in V,$$

and the regularity estimate

$$\|u\|_{H^{2+s}(\Omega)} \leq C\|f\|_{H^s(\Omega)},$$

holds for all $f \in H^s(\Omega)$. Let V_h be a finite element subspace of V satisfying

$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \leq C h^{1+s} \|u\|_{H^{2+s}(\Omega)},$$

and define $u_h \in V_h$ via

$$a(u_h, v) = (f, v) \text{ for all } v \in V_h.$$

Recall that the H^{-s} norm is defined by

$$\|u\|_{H^{-s}(\Omega)} = \sup_{0 \neq v \in H_0^s(\Omega)} \frac{(u, v)}{\|v\|_{H^s(\Omega)}}.$$

Show that

$$\|u - u_h\|_{H^{-s}(\Omega)} \leq C\, h^{1+s} \|u - u_h\|_{H^1(\Omega)}.$$