Comprehensive Exam – Numerical Analysis, August, 2016

General Instructions: Define your terminology and explain your notation. If you use a standard result, state it clealy first; otherwise, give clear and complete proofs of your claims. 4 problems completely correct guarantee a pass. Partial solutions will also be considered on their merit.

Notation: In problems 1-2, let $k > 0$ and $h > 0$ be the time step size and spatial step size, respectively. Denote by U_m^n the value of the grid function at point $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$ and $n \in \{0\} \cup \mathbb{Z}^+$.

1. Construct a finite difference scheme that is consistent with the strongly damped wave equation

$$u_{tt} - u_{txx} - u_{xx} = f.$$

   You must verify the consistency of the scheme.

2. Analyze the stability of the Crank-Nicolson scheme

$$\frac{U_m^{n+1} - U_m^n}{k} - \frac{b}{2}\left(\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2} + \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}\right) = 0$$

   for the heat equation $u_t - bu_{xx} = 0$, where b is a positive constant.

3. Consider the 2D elliptic equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u + u_x = f & \text{in } \Omega = (0,1)^2, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

   Define a grid $(x_m, y_n) = (mh, nh)$ for $0 \le m, n \le N$ and $h = \frac{1}{N}$. Show that the scheme

$$\frac{4U_{m,n} - U_{m-1,n} - U_{m,n-1} - U_{m+1,n} - U_{m,n+1}}{h^2} + \frac{U_{m,n} - U_{m-1,n}}{h} = f(x_m, y_n)$$

   satisfies the following discrete maximum principle: *if $f(x, y) \le 0$ for all $(x, y) \in \Omega$, then the maximum value of $U_{m,n}$ is attained on $\partial\Omega$, i.e.,*

$$\max_{0 \le m,n \le N} U_{m,n} = \max_{(x_m, y_n) \in \partial\Omega} U_{m,n}.$$

4. Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $f \in L^2(\Omega)$, α and β be real constants, $V = \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0\}$. Prove existence and uniqueness of $u \in V$ such that, for all $v \in V$,

$$a(u, v) := \int_\Omega (\nabla u \cdot \nabla v + \alpha u_x v + \beta u_y v)\, dxdy = F(v) := \int_\Omega fv\, dxdy.$$

5. Assume the same as in problem 4 and that V_h is a subspace of V, u_h is such that $a(u_h, v) = F(v)$ for all $v \in V_h$ and I_h is a linear mapping from $V \cap H^2(\Omega)$ to V_h such that

$$\|v - I_h v\|_{H^1(\Omega)} \le C_0 h |v|_{H^2(\Omega)}, \quad v \in V \cap H^2(\Omega),$$

   where C_0 is independent of v and h. Prove existence of a constant $C > 0$, independent of u and h, such that

$$\|u - u_h\|_{H^1(\Omega)} \le Ch|u|_{H^2(\Omega)}, \quad \|u - u_h\|_{L^2(\Omega)} \le Ch^2|u|_{H^2(\Omega)}.$$

6. Let $\Omega = (0,1)^2$, $M \ge 10$ be a positive interger and $h = 1/M$. Let $\mathcal{T} = \{K_{i,j}\}_{i,j=1}^M$, where $K_{i,j} = [(i-1)h, ih] \times [(j-1)h, jh] \subset \bar{\Omega}$. Let $\mathcal{P} = \{v \in C(\bar{\Omega}) : v(x, y) = a_0 xy + a_1 x + a_2 y + a_3,\ a_i \in \mathbb{R}, i = 0, 1, 2, 3.\}$. For each $K \in \mathcal{T}$, let $\mathcal{N} = \{N_i\}_1^4$, where $N_i(v) = v(z_i)$ for $v \in C(K)$ and z_i's are the four vertices of K. Let $\mathcal{I}_\mathcal{T}$ be the global interpolation operator for the family $\{(K, \mathcal{N}, \mathcal{P}) : \ K \in \mathcal{T}\}$ and $W_h = \{\mathcal{I}_\mathcal{T} f : \ f \in C(\bar{\Omega})\}$.

   (a) Prove $(K, \mathcal{N}, \mathcal{P})$ is a finite element for each $K \in \mathcal{T}$.

   (b) Prove $W_h \subset C(\bar{\Omega}) \cap H^1(\Omega)$.