Comprehensive Exam – Numerical Analysis, May/June 2016

General Instructions: Define your terminology and explain your notation. If you need a standard result, state it before using it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct guarantee a pass. Partial solutions will also be considered on their merit.

Notation: In problems 1-2, let $k > 0$ and $h > 0$ be the time step size and spatial step size, respectively. Denote by U_m^n the value of the grid function at point $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$ and $n \in \{0\} \cup \mathbb{Z}^+$.

1. Compute the order of accuracy of the Crank-Nicolson scheme

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1} + U_{m+1}^n - U_{m-1}^n}{4h} = \frac{f_m^{n+1} + f_m^n}{2},$$

for the one-way wave equation $u_t + au_x = f$.

2. Consider the reversed Lax-Friedrichs scheme

$$\frac{\frac{1}{2}(U_{m+1}^{n+1} + U_{m-1}^{n+1}) - U_m^n}{k} + a\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2h} = 0$$

for the one-way wave equation $u_t + au_x = 0$. Set the refinement path with $\lambda = \frac{k}{h}$ being constant. Determine a sufficient and necessary condition for the scheme to be stable.

3. Consider the 2D elliptic equation:

$$\begin{cases} u_{xx} + u_{xy} + u_{yy} = f & \text{in } \Omega = (0,1)^2, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Construct a scheme on the grid $(x_m, y_n) = (mh, nh)$, for $0 \le m, n \le N$ and $h = \frac{1}{N}$, with at least $O(h^2)$ order of accuracy. You must prove the order of accuracy of your scheme.

4. Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, Γ a connected closed proper subset of $\partial\Omega$ with positive arc-length, $V = \{v \in H^1(\Omega) \ : v|_\Gamma = 0\}$ and $f \in L^2(\Omega)$. Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$$\int_\Omega (u_x v_x + u_y v_y)\ dxdy = \int_\Omega fv\ dxdy, \quad \forall\ v \in V.$$

Derive the partial differential equation and the boundary condition satisfied by u. Justify your derivation rigorously.

5. Let K be a triangle in $\mathbb{R}^2$ with vertices z_i, $i = 1, 2, 3$. Let $\mathcal{P}$ be the space of linear functions of x, y and

$$\mathcal{N} = \{N_i \mid N_i(p) = p(z_i),\ i = 1, 2, 3;\ p \in \mathcal{P}\}.$$

   (a) State definition of a finite element and prove $(K, \mathcal{P}, \mathcal{N})$ is a finite element.

   (b) For the case $z_1 = (1, 0)$, $z_2 = (0, 1)$ and $z_3 = (0, 0)$, find the nodal basis of $\mathcal{P}$ for $(K, \mathcal{P}, \mathcal{N})$.

6. Let $(K, \mathcal{P}, \mathcal{N})$ be as given in above problem and $\mathcal{I}_K$ be the local interplation operator.

   (a) Prove $\mathcal{I}_K(v)$ is well defined for $v \in W_2^2(K)$.

   (b) Suppose $\text{diam}(K) = 1$. Prove existence of a constant $C > 0$, independent of v, such that

$$|v - \mathcal{I}_K(v)|_{W_2^k(K)} \le C|v|_{W_2^2(K)}, \quad \text{for } k = 0, 1, \text{ and } v \in W_2^2(K).$$

   A version of Bramble-Hilbert Lemma on Taylor polynomial approximation in Brenner-Scott book can be used. State this lemma clearly.