Comprehensive Exam – Numerical Analysis

January 2016

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Let $u_0 \in C^1(\mathbb{R}^1)$ be compactly supported and $u = u(x,t)$ be the solution of the following initial value problem:
$$u_t + u_x = u,\ x \in \mathbb{R}^1,\ t > 0,\ u(\cdot, 0) = u_0.$$
Consider the following numerical scheme:
$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{v_{m+1}^n - v_m^n}{h} = v_m^n,\ n \in \{0,1,2,\dots\},\ m \in \mathbb{Z},$$
$$v_m^0 = u_0(mh),\ m \in \mathbb{Z},$$
where $k, h > 0$ are the discretization sizes in t and x respectively and $\nu = kh^{-1}$ is assumed as a constant. Find the range of ν such that the scheme is convergent in the following sense:
$$\lim_{(nk,mh)\to(t,x)} |v_m^n - u(mh, nk)| = 0.$$
Justify your answer rigorously.

2. Consider the following numerical scheme for equation $u_t = u_{xx} + u_x$:
$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{v_{m-1}^n - 2v_m^n + v_{m+1}^n}{h^2} + \frac{v_{m+1}^n - v_{m-1}^n}{2h},$$
$$n \in \{0,1,2,\dots\},\ m \in \mathbb{Z},$$
where $k, h > 0$ are the discretization sizes in t and x respectively and $\mu = kh^{-2}$ is assumed as a constant. Find the range of μ such that the scheme is stable in 2-*norm*. Justify your answer rigorously.

3. Suppose $f \in C^4([0,1])$ and $u = u(x)$ solves the following problem:
$$-u''(x) + u(x) = f(x),\ x \in (0,1),\ u(0) = u(1) = 0.$$
Consider the numerical scheme:
$$-\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} + v_i = f(x_i),\ i = 1, \dots, N-1,\ x_i = ih,\ \text{and}\ v_0 = v_N = 0,$$
where $N (\geq 10)$ is an integer and $h = 1/N$. Prove the following:
   (a) for any given $f \in C^4([0,1])$, there exists a unique grid function $\{v_i\}_0^N$ solving the above numerical scheme;
   (b) there exists a constant $c > 0$ independent of h, such that
$$\max_{0 \leq i \leq N} |u(ih) - v_i| \leq ch^2.$$

4. Consider the problem $-u_{xx} + u_x + u = 0$ for $x \in (0,1)$ with non-homogeneous Dirichlet boundary conditions $u(0) = u(1) = 1$.
   (a) State rigorously a weak formulation in a properly defined space for this problem. Make sure that your weak formulation admits a unique solution by proving it. Be careful in the proof when you deal with the non-homogeneous boundary conditions.
   (b) State the P_1 (i.e., piecewise linear) continuous finite element approximation to the weak formulation. Show that the finite element problem also admits a unique solution.

5. Let $\mathcal{T}_h$ be a quasi-uniform triangular mesh with characteristic mesh size h and satisfying the minimum angle condition on a polygon Ω. Denote by V_h the space of P_1 continuous finite element on $\mathcal{T}_h$, i.e.,

$$V_h = \{v \in C^0(\Omega) \,|\, \text{such that } v \in P_1(T) \text{ on each } T \in \mathcal{T}_h\},$$

where $P_1(T)$ stands for the space of all linear polynomials on triangle T. Denote by $\mathcal{V}_h$ the set of all vertices in $\mathcal{T}_h$.

Define the following two norms on V_h,

$$\|v\|_0 = \left(\int_\Omega |v|^2 \, dx\right)^{\frac{1}{2}}, \qquad \|v\|_{0,h} = \left(h^2 \sum_{\mathbf{x} \in \mathcal{V}_h} |v(\mathbf{x})|^2\right)^{\frac{1}{2}}.$$

Show that these two norms are equivalent in the following way: there exist positive constants c and C independent of h such that

$$c\|v\|_{0,h} \le \|v\|_0 \le C\|v\|_{0,h} \qquad \text{for all } v \in V_h.$$

6. Let U_h and V_h be two finite dimensional spaces with bases $\{\phi_i,\, i = 1, 2 \ldots, m\}$ and $\{\psi_i,\, i = 1, 2 \ldots, n\}$, respectively. Denote by $\|\cdot\|_{U_h}$ and $\|\cdot\|_{V_h}$ the norms on U_h and V_h. Let

$$a(\cdot, \cdot) : \, U_h \times V_h \to \mathbb{R}$$

be a continuous bilinear form. Define matrix $A \in \mathbb{R}^{n \times m}$ by

$$A_{ij} = a(\phi_j,\, \psi_i) \qquad \text{for } 1 \le i \le n,\ 1 \le j \le m.$$

Prove that

(a) If there exists a positive constant C_1 such that

$$\sup_{v \in V_h} \frac{a(u,\, v)}{\|v\|_{V_h}} \ge C_1 \|u\|_{U_h} \qquad \text{for all } u \in U_h,$$

then $\ker(A) = \{\mathbf{0}\}$; (Hint: prove by contradiction.)

(b) If there exists a positive constant C_2 such that

$$\sup_{u \in U_h} \frac{a(u,\, v)}{\|u\|_{U_h}} \ge C_2 \|v\|_{V_h} \qquad \text{for all } v \in V_h,$$

then $rank(A) = n$. (Hint: consider the relation between A and A^T.)