Comprehensive Exam–Numerical Analysis

August 2015

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Find the order of accuracy of the scheme for equation $u_t + au_x = f$:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2 k}{2}\left(\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}\right) = f(t_n, x_m),$$
$$\text{for} \quad m \in \mathbb{Z},\ n \in \{0\} \cup \mathbb{Z}_+,$$

where $t_n = nk$, $x_m = mh$ and $h, k > 0$. Justify your answer rigorously.

2. Consider the numerical scheme for the equation $u_t = u_{xx} + u_{yy}$:

$$\frac{\tilde{v}_{i,j}^n - v_{i,j}^n}{k} = \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{h^2} + \frac{\tilde{v}_{i+1,j}^n - 2\tilde{v}_{i,j}^n + \tilde{v}_{i-1,j}^n}{h^2},$$
$$\frac{v_{i,j}^{n+1} - \tilde{v}_{i,j}^n}{k} = \frac{v_{i,j+1}^{n+1} - 2v_{i,j}^{n+1} + v_{i,j-1}^{n+1}}{h^2} - \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{h^2},$$
$$i, j \in \mathbb{Z}, \quad n \in \{0\} \cup \mathbb{Z}_+,$$

where $k, h > 0$. Find the necessary and sufficient condition on k, h such that the scheme is stable in 2-norm.

3. The nine-point Laplacian operator is defined as

$$\begin{aligned}\Delta_h U_{i,j} :=& \frac{1}{6h^2}(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1}) \\ &+ \frac{2}{3h^2}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}) \\ &- \frac{10}{3h^2}U_{i,j}, \quad i, j = 1, \ldots, m-1,\end{aligned}$$

where $m \geq 10$ is a positive integer and $h = 1/m$. Prove that Δ_h satisfies a discrete maximum principle.

4. Consider the boundary value problem:

$$-u''(x) = f(x),\ x \in (0,1), u(0) = u'(1) = 0. \qquad (1)$$

where $f \in C^0([0,1])$.

Let

$$V = \{v \in L^2(0,1) : a(v,v) < \infty, v(0) = 0\},$$

and

$$a(u,v) = \int_0^1 u'(x)v'(x)dx, \text{ and } (f,v) = \int_0^1 f(x)v(x)dx.$$

Define a weak solution $u \in V$ as follows:

$$a(u,v) = (f,v) \quad \forall v \in V. \tag{2}$$

(a) Suppose the weak solution $u \in V$ of (2) belongs to $C^2([0,1])$. Show that the weak solution u of (2) solves (1).

(b) Let $S \subset V$ be a finite dimensional subspace satisfying

$$\inf_{v \in S} \|w - v\|_E \le \epsilon \|w''\|,$$

where $\|u\|_E^2 = a(u,u)$ and $\|u\|^2 = \int_0^1 |u(x)|^2 dx$. Define $u_S \in S$ such that $a(u_S, v) = (f,v)$, $\forall v \in S$. Show that

$$\|u - u_S\| \le \epsilon \|u - u_S\|_E.$$

5. Let K be a polygon in $\mathbf{R}^2$ with $\text{diam}K = h$ and $W_p^l(K)$ be a Sobolev space, where $1 \le p \le \infty$ and $0 \le l$. Let $\mathcal{P}$ be a finite dimensional subspace of $W_p^l(K) \bigcap L_q(K)$, where $1 \le q \le \infty$. Show that there exist C independent of v such that for all $v \in \mathcal{P}$,

$$\|v\|_{W_p^l(K)} \le C h^{-l + \frac{2}{p} - \frac{2}{q}} \|v\|_{L_q(K)}$$

6. Let Ω be a convex polygonal domain in $\mathbf{R}^2$ and $V = H_0^1(\Omega)$. Let

$$a(u,v) = (\nabla u, \nabla v), \text{ for } u, v \in V.$$

Assume that there is a unique solution, u, to the variational problem

$$a(u,v) = (f,v), \text{ for all } v \in V.$$

Let V_h be a finite element subspace of V and define $u_h \in V_h$ via

$$a(u_h, v) = (f,v) \text{ for all } v \in V_h.$$

Define $\|v\|_E = a(v,v)^{1/2}$ for all $v \in V$.

(a) Show that

$$\|u_h - \chi\|_E \le \|u - \chi\|_E, \quad \forall \chi \in V_h.$$

(b) Using (a), show that

$$\|u - u_h\|_E \le 2\|u - \chi\|_E, \quad \forall \chi \in V_h.$$