Comprehensive Exam–Numerical Analysis

May 2015

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider the finite difference scheme:

$$v_m^{n+1} = \frac{1}{2}(\tilde{v}_{m+1}^n + \tilde{v}_{m-1}^n), \ \text{ where } \tilde{v}_m^n = v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n),\ m \in \mathbb{Z},\ n \in \{0\} \cup \mathbb{Z}_+.$$

Suppose that $a\lambda$ is a constant. Find the necessary and sufficient condition on $a\lambda$ such that the scheme is stable in 2-norm. Justify your answer rigorously.

2. Consider the numerical scheme

$$\frac{v_m^{n+1} - v_m^n}{k} = b\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}, \quad m \in \mathbb{Z},\ n \in \{0\} \cup \mathbb{Z}_+.$$

Suppose that $\mu = kh^{-2} \geq \mu_0 > 0$. Find the order of dissipation of the scheme.

3. The diagonal five-point discrete Laplacian operator for the grid function $U_{i,j}$ defined on a grid on unit square is given as

$$\Delta_h U_{i,j} := \frac{1}{2h^2}(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} - 4U_{i,j}),$$
$$i, j = 1, \ldots, m-1,$$

where $m \geq 10$ is a positive integer and $h = 1/m$. Prove that Δ_h satisfies a discrete maximum principle.

4. Let $0 = x_0 < x_1 < x_2 < ... < x_n = 1$ be a partition of $[0, 1]$ and let S be the linear space of functions v such that

i) $v \in C^0([0, 1])$

ii) $v_{[x_{i-1}, x_i]}$ is a linear polynomial, $i = 1, ...n$, and

iii) $v(0) = 0$.

Define

$$V = \{v \in L^2(0, 1) : a(v, v) = \int_0^1 |v'(x)|^2 dx < \infty \text{ and } v(0) = 0\},$$

with the norm $\|v\|_E = a(v, v)^{1/2}$ for $v \in V$.

Show that

$$\|u - u_I\|_E \le Ch\|u''\|_{L_2}$$

for all $u \in C^2([0,1]) \bigcap V$. Note that $u_I \in S$ is an interpolant of u satisfying $u_I(x_i) = u(x_i), i = 1, ..., n$.

5. Let $a(\cdot,\cdot)$ be the inner product for a Hilbert space V. Prove that the following two statements are equivalent for $F \in V'$, and an arbitrary (closed) subspace U of V.

   (a) $u \in U$ satisfies $a(u,v) = F(v)$ for all $v \in U$

   (b) u minimizes $\frac{1}{2}a(v,v) - F(v)$ over $v \in U$.

6. Let Ω be a convex polygonal domain in $\mathbf{R}^2$ and $V = H_0^1(\Omega)$. Let

$$a(u,v) = (\nabla u, \nabla v) + (u,v), \text{ for } u, v \in V.$$

Assume that there is a unique solution, u, to the variational problem

$$a(u,v) = (f,v), \text{ for all } v \in V,$$

and the regularity estimate

$$\|u\|_{H^2(\Omega)} \le C\,\|f\|_{L_2(\Omega)},$$

holds for all $f \in L_2(\Omega)$. Let V_h be a finite element subspace of V satisfying

$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \le C\,h\|u\|_{H^2(\Omega)},$$

and define $u_h \in V_h$ via

$$a(u_h, v) = (f,v) \text{ for all } v \in V_h.$$

Show that

$$\|u - u_h\|_{L_2(\Omega)} \le C\,h\|u - u_h\|_{H^1(\Omega)},$$

where C is a generic constant independent of h, f.