Comprehensive Exam–Numerical Analysis

January, 2015

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider the scheme

$$\frac{1}{2k}\left[(v_m^{n+1}+v_{m+1}^{n+1})-(v_m^n+v_{m+1}^n)\right]+\frac{a}{2h}\left[(v_{m+1}^{n+1}-v_m^{n+1})+(v_{m+1}^n-v_m^n)\right]=0$$

for the one-way wave equation $u_t + au_x = 0$. Here a is a real constant, v_m^n is the value of the grid function defined on $(x_m,\, t_n) = (mh,\, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Prove the scheme is consistent and is stable in 2-norm for all values of $\lambda = k/h$.

2. Consider the upwinding scheme

$$\frac{v_m^{n+1}-v_m^n}{k}+a\frac{v_m^n-v_{m-1}^n}{h}=b\frac{v_{m-1}^n-2v_m^n+v_{m+1}^n}{h^2},$$

for the convection-diffusion equation $u_t + au_x = bu_{xx}$, where $a > 0$, $b > 0$, and v_m^n is the value of the grid function defined on $(x_m,\, t_n) = (mh,\, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Derive an exact condition on a, b, k and h such that the scheme is stable in the maximum norm.

3. Consider the Poisson equation in polar coordinates:

$$-\Delta u \triangleq -\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]=f \qquad \text{for } 0<a\le r\le b \text{ and } 0<\alpha\le\theta\le\beta<2\pi,$$

with proper boundary conditions. Define a grid by $(r_i,\, \theta_j) = (a + ih_r,\, \alpha + jh_\theta)$ where h_r and h_θ are the step sizes. Compute the truncation error of the following scheme:

$$-\left[\frac{r_{i+\frac{1}{2}}(U_{i+1,j}-U_{i,j})-r_{i-\frac{1}{2}}(U_{i,j}-U_{i-1,j})}{r_i\, h_r^2}+\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{r_i^2\, h_\theta^2}\right]=F_{i,j},$$

where $U_{i,j}$ is the grid function defined on $(r_i,\, \theta_j)$, and $F_{i,j} = f(r_i, \theta_j)$. You can assume all functions are infinitely differentiable when computing the truncation error.

4. Let

$$\begin{cases} -\triangle u &= f \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{1}$$

Assume the following regularity estimate

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L_2(\Omega)}.$$

(a) Find the necessary condition for $f \in L_2(\Omega)$, where Ω is bounded polygonal domain, so that solutions to (1) exists.

(b) Consider the variational formulation of (1)

$$(\nabla u, \nabla v) = (f, v) \text{ for all } v \in V,$$

where $V = \{v \in H^1(\Omega); \int_\Omega v dx = 0\}$ and $(\cdot,\cdot)$ is the standard inner product. Let $V_h \subset V$ be an approximate space satisfying

$$\inf_{w \in V_h} \|v - w\|_{H^1(\Omega)} \le Ch\|v\|_2.$$

Let $u_h \in V_h$ be the approximate solution of u in (1) satisfying

$$(\nabla u_h, \nabla v) = (f, v), \text{ for all } v \in V_h.$$

Show that

$$\|u - u_h\|_0 \le Ch^2\|u\|_2.$$

5. Let $V = \{v \in H^1([0,1]) : v(0) = v^{'}(1) = 0\}$ and

$$g_x(t) = \begin{cases} t & t < x, \\ x & otherwise \end{cases}$$

for $x \in [0,1]$.

(a) Show that

$$(\nabla v, \nabla g_x) = v(x),$$

for all $v \in V$.

(b) Let $u \in V$, and let u_h satisfy $(\nabla(u - u_h), \nabla\psi) = 0$ for all ψ, where ψ is any piecewise linear continuous functions on the mesh $x_0 = 0 < x_1 < ...x_{n-1} < x_n = 1$ with $\psi(0) = 0$. Show that

$$(u - u_h)(x_i) = 0,$$

for all $i = 1, ..., n$.

Restatement: Let $V_h \subset V$ be the set of piecewise linear continuous functions ψ on a mesh $x_0 = 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ with $\psi(0) = 0$. Let $u \in V$ and $u_h \in V_h$ satisfy $(\nabla(u - u_h), \nabla\psi) = 0$ for all $\psi \in V_h$. Show that $(u - u_h)(x_i) = 0$ for $i = 1, 2, \ldots, n$.

6. Let K be a polygon in $\mathbf{R}^2$ with diam$K = h$, and $\mathcal{P}$ is a finite dimensional subspace of $W_p^l(K) \bigcap L_q(K)$, where $1 \le p \le \infty, 1 \le q \le \infty$ and $0 \le l$. Show that there exist C independent of v such that for all $v \in \mathcal{P}$,

$$\|v\|_{W_p^l(K)} \le Ch^{-l+\frac{n}{p}-\frac{n}{q}}\|v\|_{L_q(K)}$$