Comprehensive Exam–Numerical Analysis

August, 2014

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider the equation $u_t + u_{xxx} = f$ for $t \in (0, \infty)$ and $x \in \mathbb{R}$, with initial condition $u(0, x) = u_0(x)$ for all x. Let $k > 0$ and $h > 0$ be the time step size and spatial step size, respectively. Derive a finite difference scheme for this problem with order of accuracy at least $O(k) + O(h)$, and prove your claim.

2. Let $a > 0$ and consider the following *skewed leapfrog* method for solving the one-way wave equation $u_t + au_x = 0$:
$$v_m^{n+1} = v_{m-2}^{n-1} - (a\lambda - 1)(v_m^n - v_{m-2}^n),$$
where $\lambda = k/h$ and v_m^n is the value of the grid function defined on $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Find the range of $a\lambda$ such that the scheme is stable.

3. Consider the Lax-Wendroff scheme
$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \frac{a^2\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n)$$
for the one-way wave equation $u_t + au_x = 0$, where $\lambda = k/h$ and v_m^n is the value of the grid function defined on $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Show that the scheme is dissipative of order 4 when $0 < |a\lambda| < 1$.

4. (a) Let $\Omega = [-1, 1]$ and $f(x) = 1 - |x|$. Let $g(x) = 1$ for $x < 0$ and $g(x) = -1$ for $x > 0$. Show that $D_w f = g$, where $D_w f$ denotes the weak derivative of f.

   (b) Show that $D_w g$ does not exist.

5. Let $\Omega \subset R^n, n = 2, 3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider Poisson's equation

$$-\triangle u = f, \text{ in } \Omega$$

with Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega.$$

Assume that $f \in H^s(\Omega)$ and the regularity estimate

$$\|u\|_{H^{s+2}(\Omega)} \leq C\|f\|_{H^s(\Omega)}.$$

Let $V_h \subset V$ be the finite element approximation space and assume the approximate estimate

$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \leq Ch^{s+1}\|u\|_{H^{s+2}(\Omega)}$$

holds for some $s \geq 0$. Let $u_h \in V_h$ be the Galerkin approximate solution for u. Then, show that

$$\frac{|u - u_h|^2_{H^1(\Omega)}}{\|f\|_{H^s(\Omega)}} \leq \|u - u_h\|_{H^{-s}(\Omega)} \leq Ch^{s+1}\|u - u_h\|_{H^1(\Omega)}.$$

6. Let $\Omega \subset R^n, n = 2, 3$ be a bounded domain and $\partial\Omega$ is Lipschitz continuous. Consider Poisson's equation

$$-\triangle u = f, \text{ in } \Omega$$

with boundary conditions

$$u = g_D \text{ on } \Gamma \subset \partial\Omega \text{ and } \frac{\partial u}{\partial \nu} = g_N \text{ on } \partial\Omega \backslash \Gamma,$$

where $\text{meas}(\Gamma) \neq 0$, $g_D \in H^1(\Omega)$ and $g_N \in L^2(\partial\Omega\backslash\Gamma)$.

Let $V = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$, and let $V_h \subset V$ be the finite element approximation space. Then, the variational formulation is: find $u - g_D \in V$ such that

$$a(u, v) = (f, v) + \int_{\partial\Omega\backslash\Gamma} g_N v ds, \text{ for all } v \in V,$$

and we seek u_h such that $u_h - \mathcal{I}_h g_D \in V_h$ such that

$$a(u_h, v) = (f, v) + \int_{\partial\Omega\backslash\Gamma} g_N v ds, \text{ for all } v \in V_h,$$

where $\mathcal{I}_h g_D$ is an interpolant of g_D. Show that

$$|u - u_h|_{H^1(\Omega)} \leq C\Big(\inf_{v \in V_h} |u - g_D - v|_{H^1(\Omega)} + |g_D - \mathcal{I}^h g_D|_{H^1(\Omega)}\Big).$$