Comprehensive Exam–Numerical Analysis

June 2014

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider the Lax-Wendroff scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2 k}{2}\frac{(v_{m+1}^n - 2v_m^n + v_{m-1}^n)}{h^2} = 0$$

for the one-way wave equation $u_t + au_x = 0$. Here a is a real constant, v_m^n is the value of the grid function defined on $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Prove the scheme is dissipative of order 4 for $|a\lambda| < 1$, where $\lambda = k/h$.

2. Write the leap-frog scheme for the following problem

$$\begin{cases} u_t = u_x & \text{for } x \in \mathbb{R},\ t \in [0, \infty), \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

on the grid $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$.

   (a) Find the order of accuracy of the scheme.

   (b) Let $\lambda = k/h$ be a constant. Find the necessary and sufficient condition for the scheme to be stable in the discrete L^2 norm.

3. (a) Let $w(x)$ be a real valued smooth function on $x \in [a, b]$. Defined an uneven grid $a = x_0 < x_1 < \cdots < x_N = b$, with step sizes $h_i = x_i - x_{i-1}$. Let $h = \max_i h_i$. Show that

$$\frac{dw}{dx}(x_i) - \frac{w(x_{i+1}) - w(x_{i-1})}{h_i + h_{i+1}} = O(h),$$

   for $i = 1, \ldots, N-1$.

   (b) Derive a three-point finite difference approximation to $\frac{d^2 w}{dx^2}(x_i)$ using points x_{i-1}, x_i and x_{i+1}. Compute the truncation error.

   (c) Now consider the 2D elliptic equation

$$\begin{cases} -u_{xx} - u_{yy} - u_x = f & \text{for } 0 \le x \le 1, 0 \le y \le 1, \\ u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0 & \text{for } 0 \le x \le 1, 0 \le y \le 1. \end{cases}$$

   Define an uneven grid by $0 = x_0 < x_1 < \cdots < x_M = 1$ and $0 = y_0 < y_1 < \cdots < y_N = 1$. Denote $h_i = x_i - x_{i-1}$ and $k_j = y_j - y_{j-1}$. Derive a five-point finite difference scheme for this problem.

4. Let $a(u, v) = \int_0^1 (u'v' + u'v + uv)dx$ and $V = \{v \in W_2^1(0, 1) : v(0) = v(1) = 0\}$. Prove that $a(\cdot, \cdot)$ is coercive on V in H^1-norm.

5. Let T be an open, bounded set in $\mathbf{R}^2$ and $P_k(T)$ denotes polynomials of degree at most k. Let $u \in H^1(T)$ and

$$\inf_{\mathbf{v} \in P_k(T)^2} \int_T |\nabla u - \mathbf{v}|^2 dx = 0.$$

Show that $\nabla u \in P_k(T)$.

6. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary and $V = H_0^1(\Omega)$. Let

$$a(u, v) = (\nabla u, \nabla v) + (u, v), \text{ for } u, v \in V.$$

Assume that there is a unique solution, u, to the variational problem

$$a(u, v) = (f, v), \text{ for all } v \in V,$$

and the regularity estimate

$$\|u\|_{H^{2+i}(\Omega)} \leq C_1 \|f\|_{H^i(\Omega)} \text{ for } i = 0, 1,$$

holds for all $f \in H^i(\Omega)$ for $i = 0, 1$. Let V_h be a finite element subspace of V satisfying

$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \leq C_2 h^{1+i} \|u\|_{H^{2+i}(\Omega)}, \text{ for i=0,1,}$$

and define $u_h \in V_h$ via

$$a(u_h, v) = (f, v) \text{ for all } v \in V_h.$$

Recall that the H^{-1} norm is defined by

$$\|u\|_{H^{-1}(\Omega)} = \sup_{0 \neq v \in H_0^1(\Omega)} \frac{(u, v)}{\|v\|_{H^1(\Omega)}}.$$

Show that

$$\|u - u_h\|_{H^{-1}(\Omega)} \leq C_1 \cdot C_2 \, h^2 \|u - u_h\|_{H^1(\Omega)}.$$