Comprehensive Exam–Numerical Analysis

June 2010

General Instructions: Define your terminology and explain your notation. If you require a standard result, then state it before you use it; otherwise, give clear and complete proofs of your claims. 4 problems completely correct will guarantee a pass. Partial solutions will also be considered on their merit.

1. Consider the reverse Lax-Friedrichs scheme

$$\frac{\frac{1}{2}\left(v_{m+1}^{n+1}+v_{m-1}^{n+1}\right)-v_m^n}{k}+a\,\frac{v_{m+1}^{n+1}-v_{m-1}^{n+1}}{2h}=0$$

for the one-way wave equation $u_t + a\,u_x = 0$, where $x \in \mathbf{R}$ and $t \geq 0$. Under which condition is this scheme stable? Justify your answer.

2. The Crank-Nicolson scheme for the heat equation $u_t = b\,u_{xx}$ ($x \in \mathbf{R}$ and $t \geq 0$) is given by

$$\frac{v_m^{n+1}-v_m^n}{k}=\frac{1}{2}b\,\frac{v_{m+1}^{n+1}-2\,v_m^{n+1}+v_{m-1}^{n+1}}{h^2}+\frac{1}{2}b\,\frac{v_{m+1}^{n}-2v_m^{n}+v_{m-1}^{n}}{h^2}.$$

Show that this scheme is accurate of order 2 in time and order 2 in space.

3. Write out the matrix equation for the standard second order central difference approximation to the equation

$$u_{xx}+u_{yy}=f(x,y), \qquad (x,y)\in[0,1]\times[0,1]$$

with the boundary conditions

$$u(0,y)=1,\;\; u(1,y)=0,\;\; u_y(x,0)=u_y(x,1)=0.$$

4. Consider the boundary value problem:

$$-u'' + u = f, x \in (0,1), u(0)=u(1)=0.$$

   (a) Introduce a weak formulation of this problem in appropriate Sovolev spaces defined on the interval (0,1).

   (b) Prove the existence and uniqueness of the weak solution.

5. Let V be a Hilbert space and $a(\cdot,\cdot)$ a bounded, symmetric and coercive bilinear form on V. Given $F \in V'$, we want to find $u \in V$ such that $a(u,v) = F(v)$ for all $v \in V$. The Ritz-Galerkin approximation problem is the following: Given a finite-dimensional subspace $V_h \subset V$, find $u_h \in V_h$ such that
$$a(u_h, v) = F(v) \text{ for all } v \in V_h.$$
Show that u_h minimizes the quadratic functional
$$Q(v) = a(v,v) - 2F(v) \text{ for all } v \in V_h.$$

6. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary and $V = H_0^1(\Omega)$. Let
$$a(u,v) = (\nabla u, \nabla v) + (u,v), \text{ for } u, v \in V.$$
Assume that there is a unique solution, u, to the variational problem
$$a(u,v) = (f,v), \text{ for all } v \in V,$$
and the regularity estimate
$$\|u\|_{H^2(\Omega)} \leq C_1 \|f\|_{L_2(\Omega)}$$
holds for all $f \in L_2(\Omega)$. Let V_h be a finite element subspace of V satisfying
$$\inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \leq C_2\, h \|u\|_{H^2(\Omega)},$$
and define $u_h \in V_h$ via
$$a(u_h, v) = (f,v) \text{ for all } v \in V_h.$$
Show that
$$\|u - u_h\|_{L_2(\Omega)} \leq C_1 \cdot C_2\, h \|u - u_h\|_{H^1(\Omega)}.$$