Comprehensive Examination in Complex Analysis
June 2018

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\Omega)$ be the set of all holomorphic functions in a domain Ω.

1. Prove the Fundamental Theorem of Algebra: Suppose $n \geq 1, c_0, ..., c_n \in \mathbb{C},\ c_n \neq 0$, and set $p(z) = \sum_{j=0}^{n} c_j z^j$. Show that there exists $z \in \mathbb{C}$ with $p(z) = 0$.
2. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of entire functions and $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$. Show that $f_n' \to f'$ uniformly on compact subsets of $\mathbb{C}$.
3. Prove that
$$\lim_{R \to +\infty} \frac{1}{R} \int_{\Gamma_R} \cot z \, dz = 2i,$$
where $\Gamma_R = \partial Q_R$ is the positively oriented boundary of $Q_R = \{x + iy : 1 \leq x \leq R,\ |y| \leq R\},\ R \neq n\pi,\ n \in \mathbb{N}$.
4. Let $f, g \in H(\Omega)$ for a domain $\Omega \subset \mathbb{C}$. Suppose that $|f| = \operatorname{Re} g$ in Ω. Show that f and g are constants.
5. Suppose that $f \in H(D(0, R))$, where $D(0, R) = \{z : |z| < R\},\ R > 1$, and f satisfies
$$\frac{|f(z)|}{|z^n - 1|} \leq 1 \quad \text{for all } z \in (\partial \mathbb{D}) \setminus \{\omega_k\}_{k=1}^{n},$$
with $\{\omega_k\}_{k=1}^{n}$ being the n-th roots of unity. Prove that $|f(z)| \leq |z^n - 1|$ for all $z \in \overline{\mathbb{D}}$, and that $|f'(\omega_k)| \leq n,\ k = 1, \ldots, n$.
6. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and injective. Show that f is surjective.