Comprehensive Examination in Complex Analysis
January 2018

Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

1. Suppose $\mathbb{D}$ is the unit disk, $p \in \mathbb{D}$, and $f : \mathbb{D}\setminus\{p\} \to \mathbb{D}\setminus\{p\}$ is a holomorphic bijection. Show that f extends to a holomorphic bijection $F : \mathbb{D} \to \mathbb{D}$. Show that $F(p) = p$.

2. (i) State Morera's Theorem.
   (ii) Suppose f_n, $n \in \mathbb{N}$, are analytic in an open set G and $f_n \to f$ uniformly on compact subsets of G. Show that f is analytic in G.

3. Suppose f is entire and takes real values on a circle $\{z : |z - a| = r\}$. Prove that f is constant. Is this conclusion true if the circle is replaced by a line?

4. Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of one-to-one analytic functions that converges uniformly on compact subsets of a domain G to a function f. Show that if f is not constant, then it is one-to-one.

5. Find the Laurent series for
   $$f(z) = \frac{(z+1)}{z(z-4)^3}$$
   in the annulus $\{z : 0 < |z - 4| < 4\}$. Evaluate the integral $\displaystyle\int_{|z-4|=2} f(z)\,dz$, where the contour is run once in the positive direction.

6. Suppose that f is entire and
   $$h(z) := \frac{\sin z}{z^2 + z + 1}.$$
   Show that if $h \circ f$ is entire, then f is constant.