Comprehensive Examination in Complex Analysis
June 2017

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\Omega)$ be the set of all holomorphic functions in a domain Ω.

1. Suppose V is a connected open set and that f and $\overline{f}$ are both holomorphic in V. Show that f is constant.

2. Assume $F : [-1,1] \to \mathbb{C}$ is continuous, and define $f : \mathbb{C} \setminus [-1,1] \to \mathbb{C}$ by
$$f(z) = \int_{-1}^{1} \frac{F(t)}{z-t} dt.$$
Show that f is holomorphic in $\mathbb{C} \setminus [-1,1]$, and that $\lim_{z\to\infty} f(z) = 0$.

3. Suppose that f is entire and $|f(x+iy)| \leq e^{x^2-y^2}$ for all $x, y \in \mathbb{R}$. Prove that $f(z) = ce^{z^2}$ where $c \in \mathbb{C}$ and $|c| \leq 1$.

4. Show that every meromorphic function in $\mathbb{C}$ is a ratio of two entire functions.

5. Suppose that $f \in H(\mathbb{D} \setminus 0)$ and
$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \leq 1 \quad \text{for all } r \in (0,1).$$
Prove that f has a removable singularity at the origin.

6. Let ψ be a one-to-one conformal mapping of $\mathbb{D}$ onto the square $S = \{z = x + iy : |x| + |y| \leq 1\}$ that satisfies $\psi(0) = 0$ and $\psi'(0) > 0$. Show that $\psi(z)$ is real for real z, and $\psi(z)$ is imaginary for imaginary z.