Comprehensive Examination in Complex Analysis
January 2017

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\mathbb{D})$ be the set of all holomorphic functions in $\mathbb{D}$.

1. Suppose f is an entire function, $f'(0) = 1$, $f(k\pi) = 0$ for every integer k, and

$$|f(x+iy)| \le e^{|y|} \quad (x, y \in \mathbb{R}).$$

Show that $f(z) = \sin z$.

2. Suppose that f is holomorphic in the punctured disk $D' = \{z : 0 < |z| < 1\}$ and f has an essential singularity at the origin. Show (without using the Picard theorems) that $f(D')$ is dense in $\mathbb{C}$.

3. Let $\mathcal{F}$ be the family of all functions

$$F(z) := \int_0^1 \frac{f(x)\,dx}{x - z}, \quad z \in \mathbb{C} \setminus [0,1],$$

such that $f \in C([0,1])$ and $|f(x)| \le 1$ for all $x \in [0,1]$. Prove that $\mathcal{F}$ is a normal family in $\mathbb{C} \setminus [0,1]$. (To apply Montel's theorem, you need to explain why each $F \in \mathcal{F}$ is holomorphic in $\mathbb{C} \setminus [0,1]$.)

4. Show that for any $r \in (0,1)$ there is $N = N(r) \in \mathbb{N}$ such that the polynomial $z^n - nz - 1$ has $n-1$ roots in $A = \{z \in \mathbb{C} : r < |z| < 1/r\}$ when $n \ge N$.

5. Let $\{z_k\}_{k=1}^{n+2}$, $n \in \mathbb{N}$, be a set of distinct points in $\mathbb{C}$. Suppose that $P(z)$ is a polynomial of degree n such that $P(z_k) \ne 0$, $k = 1, \ldots, n+2$. Prove that $Q(z) = \prod_{k=1}^{n+2} (z - z_k)$ satisfies

$$\sum_{k=1}^{\overset{n+2}{\cancel{n}}} \frac{P(z_k)}{Q'(z_k)} = 0.$$

Hint: Compute $\displaystyle\int_{|z|=R} \frac{P(z)}{Q(z)}\,dz$ for large $R > 0$ by the Residue Theorem.

6. Let S be the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. Find an explicit conformal equivalence mapping S onto the unit disk. Hint: First map S onto the upper half-plane.