Comprehensive Examination in Complex Analysis
August 2016

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\Omega)$ be the set of all holomorphic functions in a domain Ω.

1. Let f be entire and assume that

$$\iint_{D_R} |f(x+iy)|\, dxdy \leq R$$

   for all sufficiently large $R > 0$, where $D_R := \{z = x+iy \in \mathbb{C} : |z| < R\}$. Show that f is identically zero. (Hint: Use polar coordinates and the mean value property.)

2. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, and the equation $f(z) = z$ has at least two solutions in $\mathbb{D}$. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

3. Suppose f and g are entire functions, and $|f(z)| < |g(z)|$ for all $z \in \mathbb{C}$ such that $|z| > 1$. Show that f/g is a rational function.

4. Let $N \in \mathbb{N}$ and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials of degree $\deg(p_n) \leq N$ for all $n \in \mathbb{N}$. Assume that this sequence converges uniformly on compact subsets of a domain G to a function f. Show that f is a polynomial of degree at most N.

5. Suppose f is entire and $|f(z) - i| > 1$ for all $z \in \mathbb{C}$. Show f is constant.

6. Let $f \in H(\mathbb{D})$ satisfy $f(iz) = -f(z)$ for all $z \in \mathbb{D}$. Show that there exists $g \in H(\mathbb{D})$ such that $f(z) = z^2 g(z^4)$.