Comprehensive Examination in Complex Analysis
May/June 2016

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\Omega)$ be the set of all holomorphic functions in a domain Ω.

1. Show that $z^5 + 6z^3 - 10$ has exactly two zeros, counting multiplicities, in the annulus $2 < |z| < 3$.

2. Let f and g be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume that both f and g have no zeros on $\overline{\mathbb{D}}$. Prove that if $|f(z)| = |g(z)|$ for all $z \in \partial\mathbb{D}$, then $f(z) = cg(z)$ in $\mathbb{D}$ for a constant c of modulus 1.

3. Let $\Omega = \mathbb{C} \setminus [0, 1]$.
   (i) Show that there exists f in $H(\Omega)$ such that $f'(z) = 1/z - 1/(z - 1)$.
   (ii) Show that there does not exist f in $H(\Omega)$ such that $f'(z) = 1/z + 1/(z - 1)$.

4. Suppose f is entire such that $\operatorname{Re} f(z) \geq \operatorname{Im} f(z)$ for all z with $|z| \geq 1$. Show that f is constant.

5. Let $\mathcal{F}$ be the set of all f in $H(\mathbb{D})$ such that
$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|dt \leq 1 \quad \text{for every } r \in (0, 1).$$
   Show that $\mathcal{F}$ is a normal family.

6. Find a conformal mapping of the domain $G = \{z \in \mathbb{C} : |z - 1| < 1 \text{ and } \operatorname{Im} z > 0\}$ onto $\mathbb{D}$.