Comprehensive Examination in Complex Analysis
January 2016

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $H(\mathbb{D})$ be the set of all holomorphic functions in $\mathbb{D}$.

1. Suppose that $u \geq 0$ is harmonic in $\mathbb{C}$. Show that u is constant.
2. Let $\Omega \subset \mathbb{C}$ be a connected open set and u be a real-valued harmonic function in Ω. Define the set $E = \{z \in \Omega : u_x(z) = u_y(z) = 0\}$. Assume that E has a limit point in Ω and show that $E = \Omega$.
3. Suppose that $\mathcal{F} \subset H(\mathbb{D})$ is a normal family. Show that $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is a normal family.
4. How many solutions (counting with multiplicities) does the equation $\sin z - z = 0$ have in $\mathbb{D}$? Find these solutions.
   Hint: Show that $|\sin z - z + z^3/3!| < e/5! < 1/3!$ for $|z| = 1$.
5. Let f be analytic in $\mathbb{D} \setminus \{0\}$, and assume that 0 is an essential singularity of f. If $M(r) := \max_{|z|=r} |f(z)|$, show that
$$\lim_{r \to 0+} r^n M(r) = \infty$$
   for all $n \in \mathbb{N}$.
6. Let $G \subset \mathbb{C}$ be a domain bounded by a simple closed contour L. Suppose that f is a non-constant holomorphic function in G, continuous on its closure. Prove that if $|f(z)| = 1$ for all $z \in L$, then f maps G onto $\mathbb{D}$.