Comprehensive Examination in Complex Analysis
August 2015

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will receive partial credit.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ in all problems.

1. Suppose that f is entire and $|f(z) - f(w)| \leq |z - w|$ for all $z, w \in \mathbb{C}$. Show that there exist constants a and b such that $f(z) = az + b$.

2. Find a Laurent series expansion of the function
$$f(z) = \frac{z^2}{z^2 - z - 2}$$
in the annulus $\{z : 1 < |z| < 2\}$.

3. Evaluate the integral using contour integration:
$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx.$$

4. Suppose that f is meromorphic in $\mathbb{C}$, and there are positive numbers C, k and R such that $|f(z)| < C|z|^k$ for all z with $|z| > R$. Prove that f is a rational function.

5. Let $\psi : \mathbb{D} \to G$ be holomorphic in $\mathbb{D}$, and let G be a simply connected domain. Suppose that $\varphi : G \to \mathbb{D}$ is a conformal equivalence such that $\varphi(\psi(0)) = 0$. Show that if $\psi'(0) = 2015$ and $\varphi'(\psi(0)) = 1/2015$, then $\psi = \varphi^{-1}$.

6. Suppose that $G \in \mathbb{C}$ is open, K is a compact subset of G, $p \in G$, and there exists $c > 0$ such that $|f(p)| \leq c \sup_{z\in K} |f(z)|$ for all holomorphic functions f in G. Show that in fact $|f(p)| \leq \sup_{z\in K} |f(z)|$ for all such f.
Hint: Consider f^n for $n = 1, 2, \ldots$.