Comprehensive Examination in Complex Analysis
May/June 2015

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will receive partial credit.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ in all problems.

1. Suppose that f is entire, and there is $M > 0$ and a polynomial p such that $|f(z)| \leq |p(z)|$ in the domain $\{z \in \mathbb{C} : |z| > M\}$. Prove that f is a polynomial.

2. Let $\mathcal{F}$ be a normal family of holomorphic functions in an open connected set $G \subset \mathbb{C}$. Show that the family $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is also normal in G. Is the converse true? (Prove or give a counterexample.)

3. Suppose that $G \subset \mathbb{C}$ is open, f is holomorphic in G, and $\gamma : [0,1] \to G$ is a smooth curve in G. Show that
$$\int_\gamma f'(z)\,dz = f(\gamma(1)) - f(\gamma(0)).$$

4. Suppose $u : \mathbb{D} \to \mathbb{R}$ is harmonic. Show that there exist constants a_n such that
$$u(re^{it}) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{int}, \quad r \in [0,1), t \in \mathbb{R}.$$

5. Let f be a non-constant analytic function in $\mathbb{D}$, which is continuous on $\mathbb{D} \cup \mathbb{T}$ with $|f(z)| = 1$ for all $z \in \mathbb{T}$. Prove that
(a) f has at least one zero in $\mathbb{D}$;
(b) f can be continued to a meromorphic function on $\mathbb{C} \cup \{\infty\}$ with at least one pole.

6. Find a conformal mapping of the domain $\{z \in \mathbb{D} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ onto $\mathbb{D}$.