Comprehensive Examination in Complex Analysis
January 2015

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ in all problems.

1. Define $F : \mathbb{R} \to \mathbb{C}$ by
$$F(a) = \int_{-\infty}^{\infty} \frac{e^{iat}}{1+t^2}\, dt.$$
   (i) Use the Residue Theorem to find $F(a)$ for $a \geq 0$.

   (ii) Use the result of part (i) to find $F(a)$ for $a < 0$.

2. Give an example of a non-constant holomorphic f in $\mathbb{D}$ with infinitely many zeroes in $\mathbb{D}$. (Hint: One of many ways to do this is to set $f(z) = \sin(g(z))$ for a suitable g.)

3. Suppose f is analytic in $\mathbb{D}$. Show that $f(iz) = if(z)$ for all $z \in \mathbb{D}$ if and only if $f(z) = zg(z^4)$ for some $g \in H(\mathbb{D})$.

4. Given a continuous function $u : \mathbb{C} \to \mathbb{R}$, let $u^{2014} + u^2$ be harmonic in $\mathbb{C}$. Show that u is identically constant.

5. Let f be holomorphic in $\mathbb{H}$, and let $|f(z)| \leq 1,\ z \in \mathbb{H}$. Find the largest possible value for $|f'(i)|$.

6. Let f be analytic in the strip $S = \{z \in \mathbb{C} : |\operatorname{Re} z| < 1\}$ and continuous on its closure. Assuming that f is real on ∂S, prove that it can be continued to an entire function F such that $F(z+4) = F(z)$ for all $z \in \mathbb{C}$.