Comprehensive Examination in Complex Analysis
August 2014

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \colon |z| < 1\}$ in all problems.

1. Let $f \colon H \to \mathbb{D}$ be holomorphic, where $H = \{x + iy \colon -\infty < x < \infty, y > 0\}$.
   (a) Show that $|f''(i)| \leq 2$.
   (b) Show that $|f'(i)| \leq 1/2$.
2. Let $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. Suppose that N is a positive integer and $c_n \in \mathbb{C}$ for $-N \leq n \leq N$, and define $\phi : \mathbb{T} \to \mathbb{C}$ by $\phi(e^{it}) = \sum_{n=-N}^{N} c_n e^{int}$. Show that there exist an entire function f_1 and a function f_2 holomorphic in $\mathbb{C} \setminus \{0\}$ such that $\phi(z) = f_1(z) + f_2(z)$ for all $z \in \mathbb{T}$.
3. Let $\Omega = \{z \in \mathbb{C} \colon |z| > 1/2\}$, and define $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$. Suppose that $f \in H(\Omega)$ and $\lim_{z \to \infty} f(z) = 0$. Show that $f(z) = \dfrac{-1}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{w - z}\, dw$ for $|z| > 1$.
4. Calculate with justification $\displaystyle\int_\gamma (1 - z)^{-1} e^{1/z}\, dz$ if $\gamma(t) = \frac{1}{2} e^{it}$ for $0 \leq t \leq 2\pi$.
5. Let f be holomorphic on $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$. Show that if $f(e^{it}) = 0$ for $0 \leq t \leq \pi$ then $f \equiv 0$.
6. Show that there exists a harmonic function u on $\mathbb{D}$ that is continuous on the closure of $\mathbb{D}$ with $u(e^{it}) = 0$ for $0 \leq t \leq \pi$ but $u \not\equiv 0$.