Comprehensive Examination in Complex Analysis
January 2014

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z : |z| < 1\}$ in all problems.

1. Let f be holomorphic on $\mathbb{D}$, and assume that $f(iz) = f(z)$ for all $z \in \mathbb{D}$. Show that there exists a function g holomorphic on $\mathbb{D}$ such that $f(z) = g(z^4)$ for all $z \in \mathbb{D}$.

2. Let f be holomorphic on $\mathbb{D}$, and assume that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ and that f has a zero of order 42 at the origin. Show that $|f(z)| \leq |z|^{42}$ for all $z \in \mathbb{D}$.

3. Let f be a function holomorphic on $\{z : |z| > R_0\}$ for some $R_0 > 0$. Define the residue of f at ∞ to be the residue at 0 of the function $F(z) = -z^{-2} f(1/z)$.

   (a) Define γ_R on $[0, 2\pi]$ by $\gamma_R(t) = Re^{it}$. Show that if $R > R_0$ then the residue of f at ∞ equals $-\frac{1}{2\pi i} \int_{\gamma_R} f(z)\, dz$.

   (b) Assume now that f is holomorphic on $\mathbb{C}$ except for finitely many singularities. Show that the sum of all of the residues of f, including the residue at ∞, is 0.

4. In this problem, if f is holomorphic near 0 we write $L(f)$ for $\lim_{n\to\infty} f^{(n)}(0)/n!$ when that limit exists.

   (a) Assume that, for some $r > 1$, f is holomorphic on $\{z : |z| < r\}$. Show that $L(f)$ exists and equals 0.

   (b) Find $L(f)$ if $f(z) = 1/(z-1)$ for $z \neq 1$.

   (c) Assume that, for some $r > 1$, f is holomorphic on $\{z : |z| < r\}$ except for a simple pole of residue α at 1. Show that $L(f)$ exists and equals $-\alpha$.

5. Suppose that $\{f_n\}$ is a sequence of functions holomorphic on $\mathbb{D}$, each f_n has no zero in $\mathbb{D}$, and $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Assume that f is not identically zero. Show that f has no zero in $\mathbb{D}$.

6. Let $\mathcal{F}$ be the set of all f holomorphic on $\mathbb{D}$ such that

   $$\iint_{\mathbb{D}} |f(x+iy)|\, dx\, dy \leq 1.$$

   Show that $\mathcal{F}$ is a normal family.
   Hint: If $z_0 \in \mathbb{D}$ and $0 < \rho < 1 - |z_0|$ then

   $$2\pi\rho|f(z_0)| \leq \rho \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt.$$

   (Why?) Hence

   $$\int_0^{1-|z_0|} 2\pi\rho|f(z_0)|\, d\rho \leq \int_0^{1-|z_0|} \rho \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt\, d\rho.$$