# Comprehensive Examination in Complex Analysis
## August 2013

**General Instructions:** Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will receive partial credit.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ in all problems.

1. Assume $f$ is analytic in the punctured disk $\mathbb{D} \setminus \{0\}$. Can $f'$ have a simple pole at the origin? Justify.

2. Let $\mathcal{F}$ be the family of holomorphic in $\mathbb{D}$ functions $f$ satisfying the condition $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Given a point $\xi \in \mathbb{D}$, prove that there is a function $\phi \in \mathcal{F}$ such that

$$\phi'(\xi) = \sup_{f \in \mathcal{F}} |f'(\xi)|.$$

3. Is it possible that a sequence of polynomials converges uniformly on $\mathbb{T}$ to the function $f(z) = \bar{z}$?

4. Suppose $u \geq 0$ is harmonic in $\mathbb{C}$. Show that $u$ is constant.

5. Suppose that $f$ is entire, $f(z + 2\pi) = f(z)$, $f(0) = f(\pi) = 0$, and $|f(x + iy)| \leq e^{|y|}$. Show that $f(z) = c\sin(z)$ for some constant $c$.

6. Let $H(\mathbb{D})$ be the set of functions holomorphic in $\mathbb{D}$. Suppose that $f \in H(\mathbb{D})$ and for every positive integer $n$ there exists $g \in H(\mathbb{D})$ with $f = g^n$. Show that there exists $h \in H(\mathbb{D})$ with $f = e^h$. Assume that f is not identically 0.