Comprehensive Examination in Complex Analysis
May 2013

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will receive partial credit.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ in all problems.

1. Suppose that f is a holomorphic function in $\mathbb{D}$ that satisfies

$$|f(1/n)| \leq 2^{-n}, \quad n \in \mathbb{N},\ n > 1.$$

   Prove that f is identically zero in $\mathbb{D}$.

2. Let $G \subset \mathbb{C}$ be a domain. Given a point $w \in G$, assume there are holomorphic bijections $f_j : G \to \mathbb{D}$ such that $f_j(w) = 0$ and $f_j'(w) > 0$ for both $j = 1, 2$. Without appealing to the Riemann Mapping Theorem, prove that $f_1 \equiv f_2$.

3. Show that $F(z) := \displaystyle\int_0^1 \frac{\sin(zt)}{t}\, dt$ is entire. Prove that

$$\lim_{n\to\infty} F^{(n)}(z) = 0, \quad z \in \mathbb{C},$$

   where convergence is uniform on compact subsets of $\mathbb{C}$.

4. Let P and Q be polynomials of exact degree n such that $|P(z)| \leq |Q(z)|,\ z \in \mathbb{T}$. Show that the same inequality holds in $\mathbb{C} \setminus \mathbb{D}$, provided all zeros of Q are contained in $\mathbb{D}$.

5. Suppose that f is an entire function and $|f(z)| \leq e^{|z|}$ for all z. Show that $|f'(z)| \leq e^{|z|+1}$.

6. Suppose that f, g are holomorphic in $\mathbb{D}$, and $g(z) = \overline{f(z)}$. Show that f is constant.