Comprehensive Examination in Complex Analysis
January 2013

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial\mathbb{D}$ in all problems.

1. Suppose that $r(z)$ is a rational function with N poles (counted according to multiplicities) that is bounded in a neighborhood of infinity. Show that $r(z)$ is holomorphic in a neighborhood of infinity, and that $r(z)$ has exactly N zeroes in the extended plane $\mathbb{C} \cup \{\infty\}$.

2. Assume f is analytic on the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. If f satisfies $|f(z)| \leq 1$ for $|z| = 1$ and $|f(z)| \leq 8$ for $|z| = 2$, then show that $|f(z)| \leq |z|^3$ for all $z \in A$.

3. Suppose that f is entire and $f(\mathbb{T}) \subset \mathbb{T}$. Prove that $f(z) = cz^n$, where $|c| = 1$ and $n \in \mathbb{N} \cup \{0\}$.

4. If u is harmonic in $\mathbb{C}$ and $u \geq 0$, then show that u is constant.

5. Suppose that $V \subset \mathbb{C}$ is a connected open set and f is holomorphic in V. Show that there exists F holomorphic in V with $F' = f$ if and only if
$$\int_\gamma f(z)\, dz = 0$$
for every piecewise-smooth closed curve $\gamma \subset V$.

6. Assume f is holomorphic in $\mathbb{D}$. Show that there exists a function g holomorphic in $\mathbb{D}$ such that $|f(z)| \leq g(|z|)$ for all $z \in \mathbb{D}$.