Comprehensive Examination in Complex Analysis
May 2012

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in all problems.

1. Find the domain of convergence for the series
$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{1+z^n}.$$
On what sets is this convergence uniform? Show that f is holomophic inside the set where the series converges.

2. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is harmonic. If u^2 is also harmonic show that u is constant.

3. Show that all zeros of the polynomials $P_n(z) = z^n - z - 1,\ n \in \mathbb{N}$, approach the unit circle as $n \to \infty$.
Hint: Find an annulus that contains all zeros of P_n.

4. Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = f'(0) = 0$. Show that $|f(z)| \le |z|^2$ for all $z \in \mathbb{D}$.

5. Suppose that f is the conformal mapping of $\mathbb{D}$ onto the interior of the ellipse $\{z = x+iy : x^2/4+y^2/9 = 1\}$ that satisfies $f(0) = 0$ and $f'(0) > 0$. Show that all coefficients in the Maclaurin expansion of f are real.

6. Use the Residue Theorem to find the value of
$$\int_{-\infty}^{\infty} \frac{dt}{e^t + e^{-t}}.$$
Hint: Integrate over the rectangle with vertices $-R,\ R,\ R+i\pi,\ -R+i\pi$.