Comprehensive Examination in Complex Analysis
January 2012

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in all problems.

1. Let $V = \{z \in \mathbb{C} : |z| < 2\}$ and $W = \{z \in \mathbb{C} : |z| > 1\}$. Suppose f is holomorphic in $V \cap W$. Show that there exist f_1 holomorphic in V and f_2 holomorphic in W such that $f(z) = f_1(z) + f_2(z), \; z \in V \cap W$.
   Hint: $V \cap W$ is an *annulus*, so ...

2. Let $\{f_n\}$ be a sequence of holomorphic in $\mathbb{D}$ functions that satisfy $|f_n(0)| \le 1$, $|f_n'(z)| \le 1, \; z \in \mathbb{D}, \; n \in \mathbb{N}$. Show that there is a subsequence of $\{f_n\}$ that converges to a function f holomorphic in $\mathbb{D}$.

3. We consider a sequence of polynomials p_n of exact degree n that converge to some function f uniformly on compact subsets of $\mathbb{D}$.
   (i) Prove that if all zeros of p_n are contained in $\{z \in \mathbb{C} : |z| < 1/2\}$, then $f \equiv 0$ in $\mathbb{D}$.
   (ii) Give an example of sequence p_n with all zeros on $\partial\mathbb{D}$ such that the limit function $f(z) \neq 0, \; z \in \mathbb{D}$.

4. Show that $G := \{z \in \mathbb{C} : |z^2 - 1| < 1\}$ is the union of two simply connected domains D_1 and D_2 whose boundaries intersect at just one point. Sketch G. Find examples of conformal mappings of D_1 and D_2 onto $\mathbb{D}$.

5. Suppose that f is entire and $|f(z) - f(w)| \le |z - w|$ for all z, w. Show that $f(z) = az + b$.

6. Let f be holomorphic in the upper half plane $\mathbb{H}$, and let f satisfy

$$\lim_{\mathbb{H} \ni z \to x} f(z) = 0$$

   for all x in a nonempty open set $U \subset \partial\mathbb{H} = \mathbb{R}$. Prove that $f \equiv 0$ in $\mathbb{H}$. Is this conclusion true if "holomorphic" is replaced with "harmonic"?
   Hint: The Schwarz Reflection Principle might be helpful here.