Comprehensive Examination in Complex Analysis
August 2011

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in all problems.

1. Let f be a holomorphic in $\mathbb{D}$ function that satisfies the conditions $f(0) = f'(0) = 0$ and $|f(z)| \le 1,\ z \in \mathbb{D}$. Prove that $|f(z)| \le |z|^2,\ z \in \mathbb{D}$.

2. Let $\mathbb{H}$ denote the upper half plane, and let $\overline{\mathbb{H}}$ be its closure. Suppose that f is continuous on $\overline{\mathbb{H}}$, and is holomorphic on $\mathbb{H}$. In addition, assume that there are constants $C > 0,\ \alpha > 0$ such that $|f(z)| \le C|z|^{-\alpha}$ whenever $z \in \overline{\mathbb{H}}$ is sufficiently large in absolute value. Show that
$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x - z}, \qquad z \in \mathbb{H}.$$
Hint: Apply the calculus of residues to a large semicircular contour in $\mathbb{H}$ whose base is raised slightly above the real axis.

3. Define a branch f_0 of $f(z) = \log \dfrac{z-i}{z-1}$ that is analytic at $z = 0$, with $f_0(0) = \pi i/2$. What is the radius of convergence for the Maclaurin series of f_0?

4. Let f be holomorphic in $\mathbb{D} \setminus \{0\}$, and assume that

No.
$$\lim_{r \to 1-} \operatorname{Im} f(re^{it}) = 0, \qquad t \in [0, 2\pi).$$
Prove that there exists a holomorphic in $\mathbb{C} \setminus \{0\}$ function F such that $F(z) = f(z),\ z \in \mathbb{D} \setminus \{0\}$, and $F(1/\bar{z}) = \overline{F(z)},\ z \in \mathbb{D} \setminus \{0\}$.
Hint: Use a conformal mapping of the upper half plane onto $\mathbb{D}$.

5. Assuming that f is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, show that f can be uniformly approximated on compact subsets of $\mathbb{D}$ by rational functions of the form
$$r(z) = \sum_{k=1}^{n} \frac{a_k}{z - z_k}, \qquad a_k \in \mathbb{C},\ |z_k| = 1,\ k = 1, \dots, n.$$

6. Suppose that f is holomorphic in a neighborhood of $\zeta \in \mathbb{C}$, with $f^{(m)}(\zeta) = 0$ for $m = 1, \dots, n-1$, and $f^{(n)}(\zeta) \ne 0$. Prove that the equation $f(z) = A$ has exactly n solutions in a neighborhood of ζ (counted according to multiplicities), if $|A - f(\zeta)| < \epsilon$ for a sufficiently small $\epsilon > 0$.