Comprehensive Examination in Complex Analysis
May 2011

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in all problems.

1. Use the calculus of residues to compute the integral
$$\int_0^\infty \frac{dx}{(x+1)\sqrt{x}}.$$

2. Construct a conformal mapping from the region $\Omega := \{z \in \mathbb{C} : |z| > 1,\ 0 < \arg(z) < \pi\}$ onto $\mathbb{D}$.

3. Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions in $\mathbb{D}$ such that $|f_n(z)| \leq 1$ for all $z \in \mathbb{D}$ and all $n \in \mathbb{N}$. Suppose that $\{f_n\}_{n=1}^\infty$ converges pointwise in $\mathbb{D}$ to a function f. Prove that f is holomorphic in $\mathbb{D}$.

4. How many zeros does $z^6 + 3z^4 + 1$ have in the upper half of $\mathbb{D}$?

5. Let f be an entire function that is real-valued on $\{z \in \mathbb{C} : |z| = 1\}$. Prove that f is constant.

6. Suppose that f is holomorphic and has no zeros in the annulus $A := \{z \in \mathbb{C} : 1/2 < |z| < 3\}$. Define
$$M_r := \max_{|z|=r} |f(z)|, \quad 1/2 < r < 3.$$
Prove that
$$|f(z)| \leq M_1 |z|^{(\log M_2 - \log M_1)/\log 2}, \quad 1 \leq |z| \leq 2.$$
Hint: Construct a harmonic function in A such that $h(z) = \log M_1$ when $|z| = 1$, and $h(z) = \log M_2$ when $|z| = 2$.