Comprehensive Examination in Complex Analysis
January 2011

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in all problems.

1. Let $U = \mathbb{C} \setminus [0, 1]$. Show that there is a holomorphic function on U such that $\exp(f(z)) = z/(1-z)$ for all $z \in U$.

2. For each natural number n, define $f_n(z) = z^n \exp(2-z) - 1$.
   (a) Show that f_n has n zeros in $\mathbb{D}$.
   (b) Show that all the zeros of f_1 in $\mathbb{D}$ are real, and have multiplicity one.

3. Let $\mathcal{C}$ be the imaginary axis traversed in the positive direction. Compute
$$\int_{\mathcal{C}} \frac{e^z \, dz}{z^2 - 4}.$$

4. Consider a class $\mathcal{F}$ of functions
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in \mathbb{C})$$
   satisfying the condition $\sum_{n=0}^{\infty} |a_n|^2 \leq 1$.
   (a) Show that every $f \in \mathcal{F}$ is holomorphic in $\mathbb{D}$.
   (b) Prove that $\mathcal{F}$ is a normal family in $\mathbb{D}$.
   (c) Show that each $f \in \mathcal{F}$ satisfies $(1 - |z|^2)|f(z)| \leq 1, \ z \in \mathbb{D}$.

5. Let f be holomorphic in $\mathbb{D} \setminus [-1/2, 1/2]$ and continuous in $\mathbb{D}$. Assuming that f is real valued on $[1/2, 1)$, prove that f is holomorphic in $\mathbb{D}$. Hint: Use Reflection Principle.

6. Let $h : \mathbb{C} \to \mathbb{R}$ be harmonic and non-constant. Show, without using the Picard Theorem, that $h(\mathbb{C}) = \mathbb{R}$.