Comprehensive Examination in Complex Analysis
Summer 2010

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Throughout the exam $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathbb{D}' = \mathbb{D} - \{0\}$.

1. Let $h : \mathbb{C} \to \mathbb{R}$ be a harmonic function and suppose that there are constants $c < 1$ and $R > 0$ such that $h(z) \leq c \log |z|$, $|z| > R$. Show that h is constant.

2. By considering an appropriate contour integral, evaluate the improper integral

$$\int_0^\infty \frac{t^\alpha}{t^2 + 1}\, dt$$

for $-1 < \alpha < 1$.

3. Suppose that f is holomorphic on $\mathbb{D}$, $|f'(z)| \leq 1$ for all $z \in \mathbb{D}$, and $f(0) = f'(0) = 0$. Show that

$$|f(z)| \leq \frac{|z|^2}{2}$$

for all $z \in \mathbb{D}$. Determine all possibilities for f if, in addition, there is some $w \in \mathbb{D}'$ such that $|f(w)| = \frac{1}{2}|w|^2$.

4. Let f be a meromorphic function on $\mathbb{D}'$ and suppose that there is a sequence $\{p_n\} \subset \mathbb{D}'$ such that $p_n \to 0$ and each p_n is a pole of f. Let $U \subset \mathbb{D}$ be an open set containing 0. Show that $f(U - \{0\})$ is dense in $\mathbb{C}$.

5. Let $M > 0$ and consider the set $\mathcal{F}$ of all functions f such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $a_n \in \mathbb{C}$ and $\sum_{n=0}^{\infty} |a_n| \leq M$.

(a) Show that every $f \in \mathcal{F}$ is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$.

(b) Show that $\mathcal{F}$ is a normal family on $\mathbb{D}$.

(c) Give an example of a sequence $\{f_k\} \subset \mathcal{F}$ that converges uniformly on each compact subset of $\mathbb{D}$, but does not converge uniformly on $\overline{\mathbb{D}}$.

6. Let

$$\mathbb{S} = \{z \in \mathbb{C} \mid -1 < \mathrm{Re}(z) < 1,\ -1 < \mathrm{Im}(z) < 1\}$$

and suppose that f is an entire function such that $f(\mathbb{S}) \subset \mathbb{S}$. Show that $|f'(0)| \leq \sqrt{2}$.