Comprehensive Examination in Complex Analysis
January 2010

General Instructions: Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Throughout the exam $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $[a, b]$ denotes the line segment from a to b.

1. Let u and v be real-valued harmonic functions on $\mathbb{D}$. Prove or give a counterexample to each statement.

(a) The function $u + iv$ is holomorphic on $\mathbb{D}$.

(b) The functions uv and u^2 are harmonic on $\mathbb{D}$.

(c) If $u + iv$ is holomorphic on $\mathbb{D}$ then $u^2 - v^2$ is harmonic on $\mathbb{D}$.

2. Let f be an entire function and suppose that $|f(x + iy)| \leq e^x$ for all $x, y \in \mathbb{R}$. Show that there is a constant $c \in \mathbb{C}$ with $|c| \leq 1$ such that $f(z) = ce^z$ for all $z \in \mathbb{C}$. Can the same conclusion be drawn if we instead assume that $|f(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$?

3. Let $T > 0$ and denote by C the rectangular contour made up of the segments $[-T, T]$, $[T, T + 2i]$, $[T + 2i, -T + 2i]$, and $[-T + 2i, -T]$.

(a) Use the theory of residues to evaluate the contour integral

$$\int_C \frac{z(2i - z)}{e^{\pi z} - e^{-\pi z}}\, dz.$$

(b) Use the result of (a) to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{t}{e^{\pi t} - e^{-\pi t}}\, dt.$$

4. Let $\lambda > 1$. Show that the equation $e^{iz} + z^2 + \lambda^2 = 0$ has a unique solution in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

5. Let $a \in \mathbb{D}$ and suppose that f is a holomorphic function on $\mathbb{D}$ such that $f(a) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Prove that

$$|f'(a)| \leq \frac{1}{1 - |a|^2}$$

and describe those f for which this inequality is an equality. [Hint: Use Schwarz's Lemma.]

6. Let $U \subset \mathbb{C}$ be a bounded, connected, non-empty, open set. Let f be a non-constant holomorphic function on U and suppose that for all p in the boundary of U

$$\lim_{z \to p,\ z \in U} |f(z)| = 1.$$

Show that $f(U) = \mathbb{D}$. [Hint: The set $\mathbb{D}$ is connected.]