**Comprehensive Examination in Complex Analysis**
**August 2009**

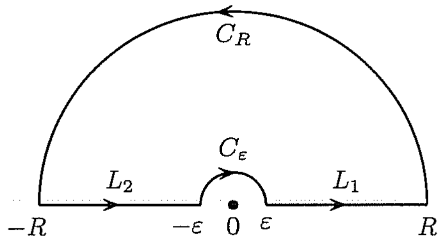
**General Instructions:** Attempt all problems. Four complete solutions will guarantee a pass. Partial solutions will be considered on their merits.

Throughout the exam, $D$ denotes the open unit disk centered at 0.

**1.** Let $\tan(z) = \sum_{n=0}^{\infty} c_n z^n$ be the Taylor series of the tangent function at 0. Show that there is a constant $K > 0$ such that $|c_n| \le K(2/3)^n$ for all $n \ge 0$.

**2.** Let $f$ be an entire function and suppose that $|f'(z)| \le |z|$ for all $z \in \mathbb{C}$. Show that there are constants $a, b \in \mathbb{C}$ with $|b| \le 1/2$ such that $f(z) = a + bz^2$.

**3.** Let $C = C_\varepsilon + L_1 + C_R + L_2$ be the contour shown in the diagram. Here $0 < \varepsilon < 1 < R$. Let $\sqrt{z}$ and $\log(z)$ be those branches of the indicated functions that are real on the positive real axis and continuous on $\mathbb{C}$ with the negative imaginary axis removed.




(a) Evaluate the contour integral

$$\int_C \frac{\sqrt{z}\log(z)}{z^2+1}\,dz$$

using the theory of residues.

(b) By considering the imaginary part of the above integral, find the value of the improper integral

$$\int_0^\infty \frac{\sqrt{t}\log(t)}{t^2+1}\,dt.$$

[You should obtain the correct answer and mention all the major steps, but it is not necessary to write out every detail.]

**4.** Let $f$ be a holomorphic function on $D$ and suppose that $0 < |f(z)| < 1$ for all $z \in D$. Prove that

$$|f(0)|^{(1+|z|)/(1-|z|)} \le |f(z)| \le |f(0)|^{(1-|z|)/(1+|z|)}$$

for all $z \in D$. [Hint: Harnack's inequality.]

**5.** Let

$$S = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1,\ \mathrm{Im}(z) > 0\}.$$

Let $f$ be a function that is continuous on the closure of $S$ (in $\mathbb{C}$) and holomorphic on $S$. Suppose that $|f(z)| \le 1$ for all $z$ in the boundary of $S$ (in $\mathbb{C}$) and that $|f(z)| \le |z|$ for all $z \in S$. Show that $|f(z)| \le 1$ for all $z \in S$. [Hint: For $\varepsilon > 0$, consider the function $g_\varepsilon(z) = f(z)e^{i\varepsilon z}$; then let $\varepsilon \to 0$.]

**6.** Let $\psi$ be a conformal bijection from $D$ to the square

$$R = \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| + |\mathrm{Im}(z)| < 1\}$$

that satisfies $\psi(0) = 0$ and $\psi'(0) > 0$. Show that $\psi(z)$ is real if $z$ is real and that $\psi(z)$ is imaginary if $z$ is imaginary. [Hint: For the first half, consider the function $\phi(z) = \overline{\psi(\bar{z})}$ and use the Riemann mapping theorem.]