Complex Analysis — January 2009

Five complete solutions will be a pass.

Notation: $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit disk; $H(\mathbb{D})$ is the set of holomorphic functions in $\mathbb{D}$.

1. (i) Suppose that $\mathcal{F}_1 \subset H(\mathbb{D})$ is a normal family, and let $\mathcal{F}_2 = \{ f \in H(\mathbb{D}) : f(0) = 0,\ f' \in \mathcal{F}_1 \}$. Show that $\mathcal{F}_2$ is a normal family.

(ii) Let $\mathcal{F}$ be the collection of all $f \in H(\mathbb{D})$ such that $f(0) = 0$, $f'(0) = 0$, and $|f''(w)| \leq 1/(1-|w|)$ for all $w \in \mathbb{D}$. Show that $\mathcal{F}$ is a normal family.

2. (i) Suppose that f is an entire function. Show that there exists an entire function g with $f = g^2$ if and only if every zero of f has even order.

(ii) Give an example of a function f holomorphic in the annulus $A = \{ z \in \mathbb{C} : \frac{1}{2} < |z| < 2 \}$ such that f has no zero in A but there does not exist $g \in H(A)$ with $f = g^2$; explain. *Hint*: If $f = g^2$ then what can you say about

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)}\, dz?$$

3. Let f be an injective conformal mapping of $\mathbb{D}$ onto a domain G that is simply connected and symmetric about the real axis (i.e., $\overline{z} \in G$ if and only if $z \in G$). Suppose that $0 \in G$ and that f satisfies $f(0) = 0$, $f'(0) > 0$. Prove that all the coefficients in the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ are real.

4. Suppose that f is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$; suppose that $I \subset \partial\mathbb{D}$ is a non-empty open arc and $f(e^{it}) = 0$ for all $e^{it} \in I$. Prove that f is identically zero. (*Note:* There are various ways to do this problem; some solutions may be simpler for the equivalent problem in the upper half-plane.)

5. Suppose that f is holomorphic in $\{ z \in \mathbb{C} : 0 < |z| < 2 \}$, $|f(z)| \leq 1$ for all z with $|z| = 1$, n is a positive integer, and there exists c such that $|f(z)| \leq c|z|^{-n}$ for all z with $0 < |z| < 1$. Show that $|f(z)| \leq |z|^{-n}$ for all z with $0 < |z| < 1$.

6. Suppose that p is a polynomial of degree n and $|p(z)| \leq 1$ for all z with $|z| = 1$. Show that $|p(z)| \leq |z|^n$ for all z with $|z| > 1$. *Hint*: You can use Problem 5 even if you haven't done that problem.