PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 2008

Five complete solutions will constitute a pass.

1. Let $n \geq 2$ be an integer.
(a) Let $P(z) = \frac{z^n - 1}{z-1}$. Find $P(1)$.
(b) Let $q_k, k = 1, 2, \ldots, n$, denote the vertices of a regular n-sided polygon inscribed in the unit circle. Let d_k be the distance between q_k and q_1. Show that $\prod_{k=2}^{n} d_k = n$.

2. Suppose that

$$\max_{|z|=1} |p_n(z)| \leq 1,$$

where p_n is a polynomial of degree at most n. Prove that

$$\max_{|z|=R} |p_n(z)| \leq R^n, \quad R \geq 1,$$

with equality if and only if $p_n(z) = cz^n$, $|c| = 1$.

3. Prove that $\sum_{n=1}^{\infty} z^{n!}$ is analytic in the unit disk, but not in any larger domain.

4. Let $f(z)$ be an entire function which is real on the real axis and purely imaginary on the imaginary axis. Prove that $f(z) = -f(-z)$ for all z.

5. Compute the following real-valued improper integral. Justify your steps.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx$$

6. Let $R > 0$. Show that there exists $N = N(R)$ such that

$$s_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!}$$

has no zeros in $\{z : |z| < R\}$ for all $n \geq N$.