Complex Exam - August 2007

Five complete solutions will count as a pass.

Note: If $n < m$ then you may use the result of problem n in your solution to problem m, even if you have not done problem n.

1. Fix an integer $n > 1$. Find the radius of convergence of the power series

$$\frac{1}{1+z^2+z^4+\cdots+z^{2n-2}} = \sum_{j=0}^{\infty} c_j(z-1)^j.$$

2. Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and f has two distinct fixed points in $\mathbb{D}$. Show that $f(z) = z$ for all $z \in \mathbb{D}$.

3. Let f be an entire function which is real on the real axis and purely imaginary on the imaginary axis. Prove that $f(z) = -f(-z)$ for all $z \in \mathbb{C}$.

In the next two problems let $A_r = \{ z \in \mathbb{C} : 0 < |z| < r \}$ be the punctured disk of radius r.

4. (i) Suppose that f is holomorphic in A_1 and there exists $r > 0$ such that $|f(z)| \leq c|z|^2$ for all $z \in A_r$. Show that f has a removable singularity at 0.

(ii) Suppose that f is holomorphic in A_1 and there exists $r > 0$ such that $|f(z)| \leq c$ for all $z \in A_r$. Show that f has a removable singularity at 0.

5. Suppose that f is holomorphic in A_1 and f has an essential singularity at 0. Suppose that $0 < r < 1$. Show that $f(A_r)$ is dense in $\mathbb{C}$. (Using the Picard Theorems is not allowed.)

6. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective (one-to-one). Show that f is a polynomial of degree 1. (For full credit you should give a solution without using the Picard Theorems; a correct solution using the Picard Theorems will be half credit.)