PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—January 2007

Five complete solutions will constitute a pass.

Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk in the plane.

1. Suppose that f is holomorphic in a neighborhood of the closed unit disk and $|f(z)| < 1$ for all z with $|z| \leq 1$. Show that there exists exactly one z in D with $f(z) = z$.

2. Suppose f is holomorphic in D.

(i) Show that

$$\frac{f(z) - f(w)}{z - w} = \int_0^1 f'\,(tz + (1-t)w)\,dt$$

for all z, w in D with $z \neq w$.

(ii) Suppose that $\mathrm{Re}(f'(z)) > 0$ for all z in D. Show that f is one-to-one in D.

3. Suppose that f is entire and $|f(z)| \leq |z|^{3/2}$ for all z in $\mathbb{C}$. Show that $f(z) = 0$ for all z.

4. Let $f_n(z)$ be a sequence of entire functions having only real zeros. Assume that $f_n(z)$ converges to $f(z)$ uniformly on compact subsets of $\mathbb{C}$ and the entire function $f(z)$ is not identically zero. Prove that $f(z)$ has only real zeros.

5. Find a Möbius transformation whose only fixed point is i and which maps 1 to ∞.

6. Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^3}\right)$$

converges uniformly on compact subsets of $\mathbb{C}$.