## Complex Analysis — August 2006

Five complete solutions will constitute a pass.

$\mathbb{D}$ is the unit disc.

**1.** Show that there exists a rational function $R(z)$ such that

$$R(z) = 2 + 6z^3 + 2z^4 + 6z^7 + 2z^8 + 6z^{11} + \dots$$

for all $z \in \mathbb{D}$. Find two polynomials $P$ and $Q$ such that $R = P/Q$, and locate the zeroes and poles of $R$.

**2.** Suppose that $(f_n)$ is a sequence of holomorphic functions in $\mathbb{D}$ and $f_n$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$. Assume that $f$ is not constant and $f(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Prove that there exists a positive integer $N$ and a sequence of points $(z_n) \subset \mathbb{D}$ such that $f_n(z_n) = 0$ for all $n \geq N$ and $\lim_{n\to\infty} z_n = z_0$.

**3.** Suppose $P$ is a polynomial of degree $n \geq 3$. Suppose that $P$ has $n$ distinct zeroes $z_1, \dots, z_n$. Use the Residue Theorem to show that

$$\sum_{j=1}^{n} \frac{z_j}{P'(z_j)} = 0.$$

**4.** Suppose that $f$ is an entire function, $f(0) = 0$, $f'(0) = \pi$, $f(z+1) = -f(z)$ for all $z \in \mathbb{C}$, and there exists $c$ such that

$$|f(x+iy)| \leq ce^{\pi|y|}$$

for all real $x$ and $y$. Show that $f(z) = \sin(\pi z)$.

**5.** Let $F$ be the family of all functions $f$ holomorphic in $\mathbb{D}$ such that $f$ has a power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

with $|c_n| \leq n^2$ for all $n$. Show that $F$ is a normal family.

**6.** Suppose that $f$ and $g$ are holomorphic in $\mathbb{D}$, and that $|f(z)| + |g(z)| = 3$ for all $z \in \mathbb{D}$. Prove that both $f$ and $g$ are constant in $\mathbb{D}$.

(One possible solution begins by choosing constants $\alpha$ and $\beta$ with $|\alpha| = |\beta| = 1$ such that $|f(0)| = \alpha f(0)$ and $|g(0)| = \beta g(0)$, then considering the function $h = \alpha f + \beta g$; it follows that $h$ is constant because ..., and that shows that $f$ and $g$ are constant because ...

Another solution uses the fact that

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it})\, dt$$

for $0 < r < 1$ (and similarly for $g$.))