Complex Analysis — January 2006

Five complete solutions will constitute a pass.

The unit disk in the plane is $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. An *entire function* is a function analytic in the entire complex plane.

1. Suppose that P is a non-constant polynomial with real coefficients and $A > 0$. Show that there are at most finitely many real numbers x such that $|P(x)| = A$.

2. Show that there does not exist an entire function f such that $f(1/n) = 1/(n+1)$ for $n = 1, 2, \ldots$

Hint: Show that there does exist exactly one such f analytic in a neighborhood of the origin (consider $g(z) = 1/f(1/z)$) but that this function is not entire.

3. Suppose that P is a polynomial of degree $n \geq 2$ with n distinct zeroes $z_1, \ldots, z_n$. Show that

$$\sum_{j=1}^{n} \frac{1}{P'(z_j)} = 0,$$

by applying the Residue Theorem to the integral of $1/P$ over an appropriate contour.

(Note that the result is false for $n = 1$, so any correct solution must use the fact that $n \geq 2$!)

4. Let $\Pi^+ = \{x + iy : y > 0\}$ and $Q = \{x + iy : x > 0, y > 0\}$ denote the upper half-plane and the first quadrant, respectively. Let $\mathbb{D}^+ = \{x + iy \in \mathbb{D} : y > 0\}$ denote the upper half of the unit disk.

(i) Give an example of a conformal equivalence mapping Π^+ onto Q.

(ii) Give an example of a conformal equivalence mapping $\mathbb{D}$ onto $\mathbb{D}^+$.

5. Suppose that f is analytic in the annulus $A = \{z : r < |z| < R\}$. Show that there exist a function g analytic in $\{z : r < |z|\}$ and a function h analytic in $\{z : |z| < R\}$ such that

$$f(z) = g(z) + h(z) \quad (z \in A).$$

Show that f extends to a continuous function on $\{z : |z| = R\}$ if and only if h extends to a continuous function on $\{z : |z| = R\}$.

6. Suppose that $(P_n)_{n=1}^{\infty}$ is a sequence of polynomials with non-negative coefficients such that $\lim_{n\to\infty} P_n(x)$ exists for every x in the interval $[0, 1]$. Show that (P_n) converges uniformly on compact subsets of $\mathbb{D}$ to a function f analytic in $\mathbb{D}$, and show that f extends to a continuous function on $\overline{\mathbb{D}}$.

Hint: Show that $\{P_n\}$ is a normal family.

(Note that the example $P_n(z) = z^n$ shows that $\lim_{n\to\infty} P_n(z)$ need not exist for $|z| = 1$.)